https://doi.org/10.1017/S1755020320000131

CALCULIZING CLASSICAL INFERENTIAL EROTETIC LOGIC

MORITZ CORDES

Abstract. This paper contributes to the calculization of evocation and erotetic implication as defined by Inferential Erotetic Logic (IEL). There is a straightforward approach to calculizing (propositional) erotetic implication which cannot be applied to evocation. First-order evocation is proven to be uncalculizable, i.e. there is no proof system, say FOE, such that for all X, Q: X evokes Q iff there is an FOE-proof for the evocation of Q by X. These results suggest a critique of the represented approaches to calculizing IEL. This critique is expanded into a programmatic reconsideration of the IEL-definitions of evocation and erotetic implication. From a different point of view these definitions should be seen as desiderata that may or may not play the role of a point of orientation when setting up "rules of asking".

§1. Introduction. Although the posing of questions plays a pivotal role in all kinds of cognitive fields, not only but especially the fields of science and humanities, logicians who study questions have yet to come to a general agreement, a "mainstream" if you will, about what apparatus may serve as a standard basis for erotetic logic[1]. According to standard presentations of how the study of questions by logicians developed, there were some pioneering efforts in the early and middle 20th century.[2] In the latter part of the 20th century more sophisticated approaches came to the fore. An incomplete list of the major approaches, excluding the more linguistically oriented ones, are: Kubiński's system ([13]), whose original publication predates the better known categorical approach by Belnap and Steel ([3]), Wiśniewski's *Inferential Erotetic Logic* ([24]), Hintikka's *Interrogative Model of Inquiry* ([11]), and Ciardelli, Groenendijk, and Roelofsen's *Inquisitive Semantics* ([6]). There is a lot to be said about each of these

1991 *Mathematics Subject Classification.* 03F99.

Key words and phrases. inferential erotetic logic, evocation, erotetic implication, axiomatization, calculization.

This paper largely profited from discussions with Szymon Chlebowski, Dorota Leszczyńska-Jasion, Jared Millson, Friedrich Reinmuth, Andrzej Wiśniewski, and the students at the chair for theoretical philosophy at the University of Greifswald. Furthermore, I have to thank the anonymous reviewers for valuable suggestions and corrections. Research on this paper was funded by the Ministry of Education, Science and Culture of Mecklenburg Vorpommern (THEORIA UG 15).

[1]The expressions 'erotetic logic' and 'question logic' will be used interchangably.

[2]This takes into account only approaches in post-Fregean frameworks. Of course, questions have been a subject of study before and some pre-Fregean approaches are very subtle. Aristotle's *Topics* are one example from antiquity. Some impressions of the role of questions within scholastic disputations can be gained from [1]. A token of modern pre-Fregean study of questions is included in the *Dianoiology* of Lambert's *New Organon* ([14, Dian., ch. 3, 7]).

approaches and, indeed, about those that I omitted, too. However, Inferential Erotetic Logic (IEL) is of special interest when it comes to defining relations between questions and declarative expressions which can be considered to be *inferential* relations.[3]

Large parts of the IEL program are outlined and developed in Andrzej Wiśniewski's 1995 book *The Posing of Questions* ([24]). In this book the author provides a formalism to represent different kinds of questions. He then proceeds to investigate, among other things, the relations of *evocation* and *erotetic implication*. Very roughly: A question is *evoked* by a set of statements iff the set guarantees that there is a true answer to the question but does not decide which one it is. When intuitively applied to natural language, one could say, for example: The set {'all philosophers are vampires or werewolves', 'Plato is a philosopher'} evokes the question 'Is Plato a vampire or a werewolf?'. A question together with a set of statements *erotetically implies* a question iff (i) any answer to the first question together with the set guarantees that there is a true answer to the second question and (ii) any answer to the second question together with the set guarantees that there is a true answer within a proper subset of the answers to the first question. The question 'Is Plato a vampire or a werewolf or a mummy?' together with {'anybody is undead iff he is a vampire or a mummy'} erotetically implies the question 'Is Plato undead or a werewolf?'.

Both relations, evocation and erotetic implication, are defined in IEL by means of a multiple conclusion entailment relation which can be defined in a number of ways.[4] For classical propositional and classical first-order IEL the usual truth-table semantics and model-theoretic semantics are often employed to define the multiple conclusion entailment relation. Given such a "semantic" account of evocation and erotetic implication, there is a natural interest in their axiomatization (cf. [15, p. 31]). However, for 20 years no such axiomatization was available. Only recently propositional evocation was axiomatized by Wiśniewski by means of a metacalculus [26] (cf. [23, Sect. 3]). Jared Millson ([19]) provided another calculus for propositional evocation, which is more general in several respects. In particular, his calculus accomodates both, propositional evocation and propositional *regular* erotetic implication, the latter of which is a transitive subrelation of propositional erotetic implication.[5] This still leaves open three (or two and a half) more axiomatization/calculization projects to be executed within IEL:

	propositional	**first-order**
Evocation	[26], [19]	unsolved
Erotetic Implication	unsolved (but solved for *regular* erotetic implication: [19])	unsolved

[3]For a general introduction to question, see [10].

[4][25, ch. 3] outlines several ways to define *admissible partitions* of the set of all wffs, which is then used to define multiple conclusion entailment. Note that the specific definition of admissible partitions in [25, Sect. 3.1.2] is defective, but the general idea is clear. Another way to define multiple conclusion entailment is presented in [27, Sect. 2.2.1].

[5]In the conclusion of [19] Millson suggests ways to modify his calculus in order to include (non-regular) erotetic implication.

Henceforth I will cease to talk of axiomatization since providing an axiomatization is only one shape which a solution to the larger problem of finding a calculus for any of the four IEL relations may take. I will talk of *calculization*, thus allowing for non-axiomatic calculi, too. Regarding the limits of what counts as a calculus I stipulate that any finite set of Turing-computable or Church-calculable rules which allow for the construction of denumerably many finite "proofs" counts as a calculus. A calculus *defines* a relation R (e.g. one of the relations in above table) iff there is a Turing-computable or Church-calculable relation r such that for all X, φ: $R(X, \varphi)$ iff there is a finite $\mathfrak{P}$ (usually a sequence of object-language expressions or a tree whose nodes are object-language expressions) and a $Y \subseteq X$ such that $\mathfrak{P}$ is constructed in accordance to the rules of the calculus and $r(\mathfrak{P}, Y, \varphi)$. The relation r is usually trivial in the sense that Y and φ are uniquely determined and can be read off of the "surface" of $\mathfrak{P}$. Proofs in sequent style calculi, for example, usually end with inscriptions of the form $\ulcorner Y \Rightarrow \varphi \urcorner$ or something similar. In other calculi the computation of Y from $\mathfrak{P}$ is slightly more complicated, calling for the identification of the non-discharged assumption lines.

As a counterexample consider the proof theory for evocation (propositional and first-order) provided by Joke Meheus.[6] Meheus speaks of a question being "finally derived" in a proof $\mathfrak{P}$ provided it will not be "marked" (i.e. withdrawn) in any extension of the proof ([18, Definition 12]). Since there are infinitely many extensions of a proof, it is not quite clear whether there exists a computable relation relating $\mathfrak{P}$ *via all of its extensions* to a premiss set and the question.[7] – The widening of the scope from axiomatizations to calculizations allows one to take into consideration additional surface features of the frameworks provided.

After this introduction, I will proceed as follows: In the second section I will present the definitions of multiple-conclusion entailment (propositional and first-order), of evocation, and of erotetic implication. Third, I will sketch Wiśniewski's axiomatization (or calculization) of propositional evocation. Fourth, with the help of a multiple-conclusion calculus I will calculize propositional erotetic implication. Fifth, the strategy will be critically reviewed, taking into account its application to first-order erotetic implication. Sixth, one can observe that the strategy is not directly applicable to evocation and that, as a main result, *first-order evocation is not calculizable*, i.e. there is no calculus or proof system, say FOE, such that for all sets X of first-order formulas and for all first-order questions Q: X evokes Q iff there is an FOE-proof for the evocation of Q by X (or by a subset of X). Seventh and finally, the results will be summarized by updating above table. The summary will also allow for some programmatic considerations.

§2. Definitions. In this section all relevant definitions from IEL are summarized. As a general reference one may regard [24] or [25]. This and the next section do not include any substantial contributions on my part.

[6] I have to thank an anonymous reviewer for pointing out to me the importance of this paper.

[7] Note that Meheus is aware of the undecidability which is at the bottom of these problems ([18, p. 139–140]). However, I think that the issue is not sufficiently discussed in Meheus's paper. I will come back to this in the last three sections.

At the outset one needs two languages, one propositional and one first-order. The propositional language is L_0^-, the first-order language is L_1^-. The vocabulary of L_0^- consists of a countably infinite number of propositional constants[8] 'p', 'q', 'r', ... and five connectives '$\neg$', '$\wedge$', '$\vee$', '$\rightarrow$', '$\leftrightarrow$'. The inductive definition of the concept of a formula runs as usual. The propositional constants are the atomic formulas and all formulas of L_0^- are considered closed. 'φ' and 'ψ' will be used as metavariables for formulas. When talking about expressions, as I just did, I will use quotation marks; when talking about schematical concatenations of expressions I will use Quine corners, e.g.: Formulas of the form $\ulcorner \varphi \wedge \psi \urcorner$ are conjunctions.

L_1^- consists of a countably infinite number of individual constants 'a', 'b', 'c', 'a_1', ..., a countably infinite number of individual variables 'x', 'y', 'z', 'x_1', ..., a countably infinite number of n-ary predicates 'P_1^n', 'P_2^n', 'P_3^n', ... for any finite $n > 0$, the same five sentential connectives as L_0^-, and two quantifiers '$\bigwedge$' and '$\bigvee$'. The term and formula definitions run as usual, as does the definition of being a closed formula. Brackets and commas are employed as usual. In slight deviation from custom, metavariables are always Greek letters (if needed with indices): 'α' for constants; 'ξ' for variables; Φ for predicates.[9]

In order to extend these languages to include questions, new expressions need to be introduced, in particular '?', '{', and '}' for the propositional case and at least '?', '{', '}', and 'S' for the first-order case. The resulting languages are $\mathrm{L}_0^?$ and $\mathrm{L}_1^?$. Questions are not formulas but form a separate category of new expressions in both languages.[10] In $\mathrm{L}_0^?$ questions are of the form $\ulcorner ?\{\varphi_1, ..., \varphi_n\} \urcorner$ with $n > 1$ and $\varphi_1, ..., \varphi_n$ pairwise distinct closed formulas. Each formula $\varphi_1, ..., \varphi_n$ and only these are the (direct) answers to the question. Thus, the natural language question 'Is Plato a vampire or is Aristotle a vampire or are they both?' can be represented in $\mathrm{L}_0^?$ by a question of the form $\ulcorner ?\{\varphi_1, \varphi_2, \varphi_3\} \urcorner$ or, more specifically, of the form $\ulcorner ?\{\varphi_1, \varphi_2, \varphi_1 \wedge \varphi_2\} \urcorner$.

In $\mathrm{L}_1^?$ questions are of the same form as in $\mathrm{L}_0^?$ (with $\varphi_1, ..., \varphi_n$ being *closed* formulas) or of the form $\ulcorner ?S(\varphi) \urcorner$ with at least one free variable in φ. A question of the latter form can be read thus: What item (or n-tuple of items, if n variables are free in φ) satisfies φ? Again, a natural-language example: 'Who is a vampire and adores the moon?' can be formalized as '$?S(P_1^1(x) \wedge P_1^2(x, a))$' or, with lower resolution, as '$?S(P_1^1(x))$'.

---

[8]What I call 'propositional constant' is often called 'propositional variable', while '$\top$' and '$\bot$' are considered propositional constants (which I would call *logical* propositional constants). I neither quantify over propositions (or their symbols) nor am I prepared, in first-order logic, to consider any non-logical predicate a variable (which, by analogy, I would have to do). Hence, I keep with 'propositional constant', just as most logicians (including me) keep with 'individual constant', although they are interpreted variably in different models.

[9]Using Greek letters as metavariables allows one to make a clear distinction between object-language expressions and schemas (i.e. sets of such expressions). E.g. 'x_i' does not serve well as a metavariable iff the object-language variables are also italic letters from the end of the alphabet.

[10]In IEL, scholars often distinguish between d-wffs (declarative well-formed formulas) and e-wffs (erotetic well-formed formulas). This corresponds to my distinction between formulas and questions.

The object-language use of curly brackets needs a bit care because the metalanguage uses these brackets, too. However, in the object languages $\mathrm{L}_0^?$ and $\mathrm{L}_1^?$ they do not have a set-theoretic meaning and '$?\{\mathsf{p},\mathsf{q}\}$' and '$?\{\mathsf{q},\mathsf{p}\}$', for example, are distinct questions, albeit with the same set of direct answers.[11] The systematic relation between the two functions of the curly brackets can be explained by defining the set of direct answers to the questions of $\mathrm{L}_0^?$ and $\mathrm{L}_1^?$. Take 'Q' to be a metavariable for questions. The sets of direct answers dQ are defined thus: If Q is a question of the form $\ulcorner ?\{\varphi_1, ..., \varphi_n\}\urcorner$, then dQ is just the set $\{\varphi_1, ..., \varphi_n\}$. If Q is a question of the form $\ulcorner ?S(\varphi)\urcorner$, then dQ is the set of all closed formulas that result from φ by substitution of individual constants for all the free variables in φ. This determines that questions of the former kind can be read as choice-questions and the questions of the latter kind as one kind[12] of wh-questions.

For the definition of evocation and erotetic implication the concept of multiple-conclusion entailment is needed. As stated above, this can be defined in different ways. On one quite common way one defines binary valuations for the propositional case ("truth tables") and a model theory for the first-order case, to the effect that by any given valuation V or any given model M, respectively, the set of all (closed) formulas is partitioned into a set of true (1) and a set of untrue (0) formulas. Presupposing that this is done in the usual (classical) way, the (single-conclusion) entailment relations can be defined thus:

Definition 2.1 (sc-Entailment). –
$X \models_P \varphi$ iff for all valuations V: if $V(\psi) = 1$ for all $\psi \in X$, then $V(\varphi) = 1$.
$X \models_{FO} \varphi$ iff for all models M: if $M \models \psi$ for all $\psi \in X$, then $M \models \varphi$.

The intended corresponding multiple-conclusion entailment relations allow for sets of formulas on the righthand side of the turnstile. The idea is that in multiple-conclusion entailment the truth of all antecedent formulas (relative to any valuation/model) guarantees the truth of at least one of the succedent formulas (relative to this valuation/model), but different valuations/models that satisfy all antecedent formulas may satisfy different succedent formulas. Thus, the antecedent formulas do not necessarily determine (the relative truth of) any of the succedent formulas.

Definition 2.2 (mc-Entailment). –
$X \Vdash\!\models_P Y$ iff for all valuations V: if $V(\psi) = 1$ for all $\psi \in X$, then there is a $\varphi \in Y$ such that $V(\varphi) = 1$.
$X \Vdash\!\models_{FO} Y$ iff for all models M: if $M \models \psi$ for all $\psi \in X$, then there is a $\varphi \in Y$ such that $M \models \varphi$.

[11] In other words: The equivalence of '$?\{\mathsf{p},\mathsf{q}\}$' and '$?\{\mathsf{p},\mathsf{q}\}$' is a matter of further regulation, e.g. by a "semantics" or by rules like the rule R4 below (Sect. 3).

[12] Note that none of the following formulas is a (direct) answer to the wh-question '$?S(P_1^1(x))$' from the preceding paragraph: '$\neg P_1^1(a)$', '$P_1^1(a) \wedge P_1^1(b)$', '$\bigwedge x\ P_1^1(x)$', '$\bigwedge x\ (P_1^1(x) \leftrightarrow x = a)$'. In order to *systematically* include such answers, one can introduce operators similar to 'S'. This then yields the *other kinds* of questions implicitly alluded to in the main text by the phrase 'one kind of'.

In the remainder of the text, the turnstile subscript 'P' (propositional) or 'FO' (first-order) will be omitted, provided both systems are concerned or context determines which relation is signified. In the informal parts, the distinction between single-conlcusion and multiple-conclusion entailment (sc- and mc-entailment) will often be left implicit.

Note that it does not hold in general that, if $X \Vvdash \{\varphi, \psi\}$, then $X \models \varphi$ or $X \models \psi$. (The converse *does* hold.) For example, $\{$'$\mathsf{p} \vee \mathsf{q}$'$\} \Vvdash \{$'$\mathsf{p}$','$\mathsf{q}$'$\}$, but $\{$'$\mathsf{p} \vee \mathsf{q}$'$\} \not\models$ 'p' and $\{$'$\mathsf{p} \vee \mathsf{q}$'$\} \not\models$ 'q'. However, for all X, φ: $X \Vvdash \{\varphi\}$ iff $X \models \varphi$. And due to the classical semantics, for all X, φ, ψ: $X \Vvdash \{\varphi, \psi\}$ iff $X \models \ulcorner \varphi \vee \psi \urcorner$. – Now evocation and erotetic implication can be defined. We begin with evocation:

Definition 2.3 (Evocation). –
$E(X, Q)$ iff $X \Vvdash dQ$ and there is no $\varphi \in dQ$ such that $X \models \varphi$.

Informally: A set of formulas evokes a question iff the set entails the question's set of direct answers but not any *single* direct answer. – Next comes erotetic implication:

Definition 2.4 (Erotetic Implication). –
$Im(Q_1, X, Q_2)$ iff
(i) for all $\varphi \in dQ_1 : X \cup \{\varphi\} \Vvdash dQ_2$ and
(ii) for all $\psi \in dQ_2$ there is $Y \subset dQ_1, Y \neq \emptyset$ such that: $X \cup \{\psi\} \Vvdash Y$.

Informally: A question together with a set of formulas erotetically implies another question iff that set (i) together with any answer to the first question entails the set of answers to the second question and (ii) together with any answer to the second question entails a non-empty *proper* subset of the set of answers to the first question. The second criterion says, in a way, that any answer to the second question downsizes the set of answers to the original question.

§3. Wiśniewski's Axiomatization of Propositional Evocation. Andrzej Wiśniewski provides a calculus $\mathrm{PMC_E}$[13] and proves soundness and completeness of the calculus with respect to evocation as defined above ([26]). The calculus is a sequent caluculus operating on *erotetic sequents* (*e-sequents*), i.e. expressions of the form $\ulcorner X \vdash Q \urcorner$ where X refers to a *finite* set of formulas and Q is a question of the form $\ulcorner ?\{\varphi_1, ..., \varphi_n\} \urcorner$. '$\vdash$' is a new expression which can be considered a "syntactic representation" of evocation. Obviously, e-sequents are not $\mathrm{L}^?_0$-expressions. Thus, $\mathrm{PMC_E}$ could be understood as a metacalculus, i.e. a metalanguage calculus.[14]

Wiśniewski's calculus consists of one schematic axiom and four transformation rules. For the axioms some preliminary terminology is needed: Any propositional constant and the negation of any propositional constant is a *literal*. Two literals

[13]'PMC' stands for 'proper mc-entailment for classical propositional logic', which is, bluntly put, evocation without the question mark. Cf. [23, Sect. 1.5]. The subscript 'E' refers to evocation.

[14]Of course, it is possible to include signs like '$\vdash$' in the vocabulary of an object language and to interpret them as an object language representation of consequence or entailment. See for example [25, Sect. 2.4.4].

are *complementary* iff one is the negation of the other. Any literal and the n-ary disjunction $\ulcorner\varphi_1 \vee ... \vee \varphi_n\urcorner$ of any literals $\varphi_1, ..., \varphi_n$ is a *clause*. – In what follows, it is convenient to neglect disjunction brackets. We will treat $\ulcorner(\varphi_1 \vee \varphi_2) \vee \varphi_3\urcorner$ and $\ulcorner\varphi_1 \vee (\varphi_2 \vee \varphi_3)\urcorner$ as the *same* formula and will usually omit the brackets.

Now the axioms of $\mathrm{PMC_E}$ have the form

$$\vdash ?\{\varphi_1, ..., \varphi_n\}$$

where $n > 1$ and any φ_i $(1 \leq i \leq n)$ is a clause of which no two disjuncts are complementary, but $\ulcorner\varphi_1 \vee ... \vee \varphi_n\urcorner$ is a clause of which at least two disjuncts are complementary. – The rules of $\mathrm{PMC_E}$ are the following:

$$\text{R1:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1\} \qquad X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_2\}}{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1 \wedge \psi_2\}} \text{ provided } \ulcorner\psi_1 \wedge \psi_2\urcorner \neq \varphi_i \text{ for any } i \leq n$$

$$\text{R2:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1\}}{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_2\}} \text{ provided } \ulcorner\psi_1 \leftrightarrow \psi_2\urcorner \text{ is logically true and } \psi_2 \neq \varphi_i \text{ for any } i \leq n$$

$$\text{R3:}\frac{X \vdash ?\{\psi \to \varphi_1, ..., \psi \to \varphi_n\}}{X \cup \{\psi\} \vdash ?\{\varphi_1, ..., \varphi_n\}}$$

$$\text{R4:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n\}}{X \vdash ?\{\psi_1, ..., \psi_n\}} \text{ provided } d(\ulcorner ?\{\varphi_1, ..., \varphi_n\}\urcorner) = d(\ulcorner ?\{\psi_1, ..., \psi_n\}\urcorner)$$

A $\mathrm{PMC_E}$-*proof* of an e-sequent is a finite labeled tree which is in accord with the rules R1, R2, R3, R4 and whose leaves are labeled with $\mathrm{PMC_E}$-axioms and whose root is the e-sequent in question. Here is an example $\mathrm{PMC_E}$-proof of '$\{\mathsf{p}, \mathsf{p} \to (\mathsf{q} \vee \mathsf{r})\} \vdash ?\{\mathsf{q}, \mathsf{r}\}$':

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdash ?\{\neg\mathsf{p} \vee \neg\mathsf{q} \vee \mathsf{r}, \neg\mathsf{p} \vee \mathsf{q} \vee \neg\mathsf{r}\} \quad \text{Ax}}{\vdash ?\{\neg\mathsf{p} \vee \neg\mathsf{q} \vee \mathsf{r}, \mathsf{p} \to ((\mathsf{p} \to (\mathsf{q} \vee \mathsf{r})) \to \mathsf{q})\}} \text{R2}}{\vdash ?\{\mathsf{p} \to ((\mathsf{p} \to (\mathsf{q} \vee \mathsf{r})) \to \mathsf{q}), \neg\mathsf{p} \vee \neg\mathsf{q} \vee \mathsf{r}\}} \text{R4}}{\vdash ?\{\mathsf{p} \to ((\mathsf{p} \to (\mathsf{q} \vee \mathsf{r})) \to \mathsf{q}), \mathsf{p} \to ((\mathsf{p} \to (\mathsf{q} \vee \mathsf{r})) \to \mathsf{r})\}} \text{R2}}{\{\mathsf{p}\} \vdash ?\{(\mathsf{p} \to (\mathsf{q} \vee \mathsf{r})) \to \mathsf{q}, (\mathsf{p} \to (\mathsf{q} \vee \mathsf{r})) \to \mathsf{r}\}} \text{R3}}{\{\mathsf{p}, \mathsf{p} \to (\mathsf{q} \vee \mathsf{r})\} \vdash ?\{\mathsf{q}, \mathsf{r}\}} \text{R3}$$

Those e-sequents of which there is a $\mathrm{PMC_E}$-proof are $\mathrm{PMC_E}$-*provable*. Wiśniewski goes on to prove soundness and completeness relative to evocation for finite antecedents ([26, Sect. 4]). This adequacy result could be formulated thus:

THEOREM 3.1 (Finite $\mathrm{PMC_E}$-adequacy w.r.t. propositional evocation). – *For all finite sets X of $L^?_0$-formulas and all $L^?_0$-questions Q: $\ulcorner X \vdash Q\urcorner$ is PMC_E-provable iff $E(X, Q)$.*

The finiteness of X is an essential limitation because evocation is not monotonic, although some infinite sets evoke some questions. Therefore, even if e-sequents were allowed to have infinte antecedents, one could not define their provability by the provability of some finite e-sequent whose antecedent is a subset of the infinite e-sequent's antecedent.[15]

[15] An anonymous reviewer relates theorem 3.1 to the compactness of propositional evocation: If $E(X, Q)$, then there is a finite $Y \subseteq X$ such that $E(Y, Q)$. With compactness one can

With respect to the language within which the questions are formulated, i.e. with respect to $L^?_0$, PMC_E is a metacalculus. The sequents it operates on contain a sequent sign and, in the antecedent, they contain curly brackets with their set-theoretic meaning and hence are not formulated in $L^?_0$. In proof-theoretical contexts this is usually considered unproblematic. Nonetheless, it is worthwhile to wonder what a pure $L^?_0$-object-language calculus for evocation would be like. This desideratum could be framed thus: Is there a classically adequate propositional evocation calculus whose "derivations" are sequences (or trees) of $L^?_0$-questions and $L^?_0$-formulas? This can be answered in the affirmative, although the presentation of my own attempt at such a calculus is left for another occasion. (Not surprisingly, it is an adaptation of Wiśniewski's metacalculus.)

§4. The Obvious Approach to Calculizing Erotetic Implication. The axiomatization of evocation as proposed by Wiśniewski is not in *obvious* agreement with the definition of evocation; it is not the case that one part of a PMC_E-proof ensures the first condition of the definition of evocation and another part of a PMC_E-proof ensures the second condition. This is not a flaw of the calculus. In fact, the superficial disparity of PMC_E and the definition of the semantic relation it calculizes, makes Wiśniewski's adequacy a substantial theorem and not just an obvious corollary.[16] With erotetic implication a more direct, or even *obvious* way of translating its definition into a calculization is available.

Informally, let us call a calculization of a "semantic" relation like erotetic implication or evocation *obvious*, if it results from the following two-step procedure: First, any conditions of sc- or mc-entailment in the definition of the semantic relation is taken to refer to the existence of a suitable derivation in a calculus equivalent to this sc- or mc-entailment relation. The calculus employed in this step can be called the auxiliary calculus. Second, the derivations of the target calculus are defined as arrangements of derivations of the auxiliary calculus. These arrangements have to be in accord with the definition of the semantic relation (e.g. evocation or erotetic implication).

---

reformulate the theorem to include infinite sets in at least two ways. (i) For all sets X of $L^?_0$-formulas (including infinite sets) and for all $L^?_0$-questions Q: if $E(X, Q)$, then there is $Y \subseteq X$ such that $\ulcorner Y \vdash Q \urcorner$ is PMC_E-provable. The finiteness of Y is determined by the role of Y as an antecedent in a provable e-sequent. This version gives up on the left-to-right direction of theorem 3.1.) – (ii) For all sets X of $L^?_0$-formulas and all $L^?_0$-questions Q there is $Y \subseteq X$ such that: $\ulcorner Y \vdash Q \urcorner$ is PMC_E-provable iff $E(X, Q)$. This version may be more interesting but neither does it say which e-sequent needs to be proven in order to say that a given infinte set X evokes a given question Q nor does it provide a rule which determines *all* Q-evoking supersets X of a given Y for which $\ulcorner Y \vdash Q \urcorner$ is provable. – If, alternatively, one does not want to rely on compactness, then only the left-to-right direction holds for infinite X. This is exactly what Wiśniewski proves for soundness ([26, theorem 8]). That this generalization holds is due to the fact that the left side of theorem 1 is only satisfiable by finite X.

[16] Another interesting feature of PMC_E is that it has a subcalculus which is a negative sc-entailment calculus, i.e. a calculus adequate with respect to $\not\models$ with finite antecedent sets. The following result is easily provable: If $X \cup \{\varphi\}$ is a finite set of formulas and ψ is a propositional constant and ψ does not occur in any element of $X \cup \{\varphi\}$, then $X \not\models \varphi$ iff $\ulcorner X \vdash ?\{\varphi, \psi, \neg\psi\} \urcorner$ is PMC_E-provable. Of course, with finite X such a "new" propositional constant ψ is always available. The proof is analogous to the core part of the proof given in Sect. 6 below.

This is a rather vague description. What is meant by it can be illustrated by an "obvious calculization" of IEL's erotetic implication. Recall that erotetic implication is defined thus:

$Im(Q_1, X, Q_2)$ iff
(i) for all $\varphi \in dQ_1 : X \cup \{\varphi\} \Vert\!\!= dQ_2$ and
(ii) for all $\psi \in dQ_2$ there is $Y \subset dQ_1, Y \neq \emptyset$ such that: $X \cup \{\psi\} \Vert\!\!= Y$.

The two parts of the definiens both state certain *positive* entailments. In other words, in order to show that erotetic implication obtains in a given case it suffices to verify these positive entailments within a suitable mc-calculus. More explicitly, in order to verify clause (i) of the definition one has to present for each answer φ to the implying question a multiple-conclusion derivation of the set of answers to the implied question from $X \cup \{\varphi\}$. In order to verify clause (ii) one has to present for each answer ψ to the implied question a multiple-conclusion derivation of a non-empty proper subset of the set of answers to the implying question from $X \cup \{\psi\}$. Roughly, if (a) there are only finitely many answers to both questions, as is the case in $L_0^?$, and (b) we have at hand a classical multiple-conclusion calculus, then all one needs to do is to arrange the relevant derivations in this mc-calculus in a suitable way. The resulting complex constitutes a derivation in the wanted target calculus.

Auxiliary calculi of the required kind (classical propositional mc-calculi) do exist. One is Gentzen's LK ([9]) limited to propositional logic (but an axiomatic calculus or a Jaśkowski style ND calculus would do, too). An easy way to apply LK to the problem at hand is to transform Gentzen's sequents into a liberalized version of Wiśniewski's e-sequents, say e*-sequents. These sequents are just like e-sequents except that they allow for questions which have only one answer $\ulcorner ?\{\varphi\}\urcorner$ or even none '?{}' or which have multiple occurences of the same answer like '$?\{\mathsf{p}, \mathsf{p}, \mathsf{q}\}$'.

Definition 4.1 (e*-sequent). –
An e*-sequent is an expression of the form $\ulcorner X \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$ where X refers to a finite (possibly empty) set of closed formulas of L_0^- and $n \geq 0$ and for all i with $1 \leq i \leq n : \varphi_i$ is a closed formula of L_0^-.

Note that it may be the case that $n = 0$ or $n = 1$ and that $\varphi_i = \varphi_j$ while $i \neq j$. This is in contrast to Wiśniewski's e-sequents [26, Sect. 2.1.3]. – Gentzen's rules will now display thus:

$$\text{AX:}\frac{}{\{\varphi\} \vdash ?\{\varphi\}} \qquad \text{DLA:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n\}}{X \cup \{\psi\} \vdash ?\{\varphi_1, ..., \varphi_n\}}$$

$$\text{DLC:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n\}}{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi\}} \qquad \text{CT:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n, \varphi_n\}}{X \vdash ?\{\varphi_1, ..., \varphi_n\}}$$

$$\text{SW:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_k, \varphi_{k+1}, ..., \varphi_n\}}{X \vdash ?\{\varphi_1, ..., \varphi_{k+1}, \varphi_k, ..., \varphi_n\}}$$

$$\text{CIA1:}\frac{X \cup \{\psi_1\} \vdash ?\{\varphi_1, ..., \varphi_n\}}{X \cup \{\psi_1 \wedge \psi_2\} \vdash ?\{\varphi_1, ..., \varphi_n\}} \qquad \text{CIA2:}\frac{X \cup \{\psi_1\} \vdash ?\{\varphi_1, ..., \varphi_n\}}{X \cup \{\psi_2 \wedge \psi_1\} \vdash ?\{\varphi_1, ..., \varphi_n\}}$$

$$\text{CIC:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1\} \qquad X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_2\}}{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1 \wedge \psi_2\}}$$

$$\text{DIA:}\frac{X \cup \{\psi_1\} \vdash ?\{\varphi_1, ..., \varphi_n\} \qquad X \cup \{\psi_2\} \vdash ?\{\varphi_1, ..., \varphi_n\}}{X \cup \{\psi_1 \vee \psi_2\} \vdash ?\{\varphi_1, ..., \varphi_n\}}$$

$$\text{DIC1:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1\}}{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1 \vee \psi_2\}} \qquad \text{DIC2:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1\}}{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_2 \vee \psi_1\}}$$

$$\text{NIA:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi\}}{X \cup \{\neg\psi\} \vdash ?\{\varphi_1, ..., \varphi_n\}} \qquad \text{NIC:}\frac{X \cup \{\psi\} \vdash ?\{\varphi_1, ..., \varphi_n\}}{X \vdash ?\{\varphi_1, ..., \varphi_n, \neg\psi\}}$$

$$\text{CdIA:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n, \varphi_{n+1}\} \qquad Y \cup \{\psi_{m+1}\} \vdash ?\{\psi_1, ..., \psi_m\}}{X \cup Y \cup \{\varphi_{n+1} \rightarrow \psi_{m+1}\} \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1, ..., \psi_m\}}$$

$$\text{CdIC:}\frac{X \cup \{\psi_1\} \vdash ?\{\varphi_1, ..., \varphi_n, \psi_2\}}{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1 \rightarrow \psi_2\}}$$

$$\text{BIA1:}\frac{X \cup \{\psi_1 \rightarrow \psi_2\} \vdash ?\{\varphi_1, ..., \varphi_n\}}{X \cup \{\psi_1 \leftrightarrow \psi_2\} \vdash ?\{\varphi_1, ..., \varphi_n\}} \qquad \text{BIA2:}\frac{X \cup \{\psi_1 \rightarrow \psi_2\} \vdash ?\{\varphi_1, ..., \varphi_n\}}{X \cup \{\psi_2 \leftrightarrow \psi_1\} \vdash ?\{\varphi_1, ..., \varphi_n\}}$$

$$\text{BIC:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1 \rightarrow \psi_2\} \qquad X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_2 \rightarrow \psi_1\}}{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1 \leftrightarrow \psi_2\}}$$

Gentzen's "Anfangssequenzen" (axioms) are here represented as the no-premiss rule AX. The rules BIA1, BIA2, and BIC are not Gentzen's. Nonetheless they are included here, because $L_0^?$ includes '$\leftrightarrow$' as a connective. Since e-sequents and e*-sequents have names for sets in their antecedents the structural rules of switch and contraction in the antecedent are superfluous and are not displayed here.

This adaptation of LK shall be named '$LK_?$'.[17] An $LK_?$-*proof* of an e*-sequent is a finite tree labeled with e*-sequents which is in accord with above rules and whose root is the e*-sequent in question. Those e*-sequents of which there is an $LK_?$-proof are $LK_?$-*provable.* Now it is clear that the following adequacy holds:

THEOREM 4.2 (Finite $LK_?$-adequacy w.r.t. mc-entailment). –
For all finite sets X of $L_0^?$-formulas and all $L_0^?$-formulas $\varphi_1, ..., \varphi_n$:
$\ulcorner X \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$ is $LK_?$-provable iff $X \Vdash_P \{\varphi_1, ..., \varphi_n\}$.

PROOF. The left-to-right direction is proven by induction on the height of the proof. *Base:* An $LK_?$-proof of $\ulcorner X \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$ has only one node (height 0), only if $n = 1$ and $X = \{\varphi_1\}$. But then, trivially, $X \Vdash_P \{\varphi_1, ..., \varphi_n\}$. *Step:* If the e*-sequent in question $\ulcorner X \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$ has an $LK_?$-proof of height $i > 0$, then the premisses of the last node in that proof have $LK_?$-proofs of a height less than i and the claim holds for these e*-sequents provable in fewer steps. Let us consider only one out of 17 ways how the last line could have been inferred (DIA): The premisses of said e*-sequent are of the form described in DIA and so is the e*-sequent in question itself as the conclusion. Thus, with

[17] Note that Millson's $LK^?$ and my $LK_?$ are entirely different calculi. Both names were chosen in a state of mutual ignorance.

$X = Y \cup \{\ulcorner\psi_1 \vee \psi_2\urcorner\}$, $\ulcorner Y \cup \{\psi_1\} \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$, $\ulcorner Y \cup \{\psi_2\} \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$, and $\ulcorner Y \cup \{\psi_1 \vee \psi_2\} \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$ are $\mathrm{LK}_?$-provable and, hence, by induction assumption, $Y \cup \{\psi_1\} \Vdash\!\models_P \{\varphi_1, ..., \varphi_n\}$ and $Y \cup \{\psi_2\} \Vdash\!\models_P \{\varphi_1, ..., \varphi_n\}$. Now let V be a valuation such that $V(\psi) = 1$ for all $\psi \in Y \cup \{\ulcorner\psi_1 \vee \psi_2\urcorner\} = X$. Thus $V(\psi) = 1$ for all $\psi \in Y$ and $V(\ulcorner\psi_1 \vee \psi_2\urcorner) = 1$ and, consequently, $V(\psi_1) = 1$ or $V(\psi_2) = 1$. Thus, $V(\psi) = 1$ for all $\psi \in Y \cup \{\psi_1\}$ or for all $\psi \in Y \cup \{\psi_2\}$. Thus, by the entailments inferred by the induction assumption, there is at least one $\psi \in \{\varphi_1, ..., \varphi_n\}$ such that $V(\psi) = 1$. Thus, $X \Vdash\!\models_P \{\varphi_1, ..., \varphi_n\}$. The other 16 cases are analogous.

The right-to-left direction is proven by induction on the total number of occurrences of connectives in the elements of $X \cup \{\varphi_1, ..., \varphi_n\}$. *Base:* If this number is zero, then all elements of $X \cup \{\varphi_1, ..., \varphi_n\}$ are propositional constants. Then $X \Vdash\!\models_P \{\varphi_1, ..., \varphi_n\}$ can only be the case, if $X \cap \{\varphi_1, ..., \varphi_n\} \neq \emptyset$. Then the e*-sequent $\ulcorner X \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$ can be proven solely by AX, DLA, DLC, and SW. *Step:* If there are occurrences of connectives in elements of $X \cup \{\varphi_1, ..., \varphi_n\}$, then at least one of the five connectives occurs in an element of X or in some $\varphi_k \in \{\varphi_1, ..., \varphi_n\}$. This leaves ten cases; I will cover three of them ($\ulcorner\neg\psi_1\urcorner \in X$, $\ulcorner\psi_1 \leftrightarrow \psi_2\urcorner \in X$, and $\varphi_k = \ulcorner\psi_1 \wedge \psi_2\urcorner \in \{\varphi_1, ..., \varphi_n\}$). Suppose $X \Vdash\!\models_P \{\varphi_1, ..., \varphi_n\}$, i.e. for all valuations V: if $V(\psi) = 1$ for all $\psi \in X$, then there is $\psi \in \{\varphi_1, ..., \varphi_n\}$ such that $V(\psi) = 1$.

(i) Suppose $\ulcorner\neg\psi_1\urcorner \in X$. Suppose $V(\psi) = 1$ for all $\psi \in X\backslash\{\ulcorner\neg\psi_1\urcorner\}$. Then $V(\psi_1) = 1$ or $V(\psi_1) = 0$. If the former, then there is $\psi \in \{\varphi_1, ..., \varphi_n, \psi_1\}$ such that $V(\psi) = 1$. If the latter, then $V(\ulcorner\neg\psi_1\urcorner) = 1$. Thus, $V(\psi) = 1$ for all $\psi \in X$. Thus, there is $\psi \in \{\varphi_1, ..., \varphi_n\} \subseteq \{\varphi_1, ..., \varphi_n, \psi_1\}$ such that $V(\psi) = 1$. Thus $X\backslash\{\ulcorner\neg\psi_1\urcorner\} \Vdash\!\models_P \{\varphi_1, ..., \varphi_n, \psi_1\}$. By induction assumption, $\ulcorner X\backslash\{\neg\psi_1\} \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1\}\urcorner$ is $\mathrm{LK}_?$-provable. NIA then proves $\ulcorner X \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$.

(ii) Suppose $\ulcorner\psi_1 \leftrightarrow \psi_2\urcorner \in X$. In order to prove $\ulcorner X \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$ we first prove that four premisses are $\mathrm{LK}_?$-provable:

1. $\ulcorner X\backslash\{\psi_1 \leftrightarrow \psi_2\} \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1, \psi_2\}\urcorner$
2. $\ulcorner (X\backslash\{\psi_1 \leftrightarrow \psi_2\}) \cup \{\psi_1\} \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1\}\urcorner$
3. $\ulcorner (X\backslash\{\psi_1 \leftrightarrow \psi_2\}) \cup \{\psi_2\} \vdash ?\{\varphi_1, ..., \varphi_n, \psi_2\}\urcorner$
4. $\ulcorner (X\backslash\{\psi_1 \leftrightarrow \psi_2\}) \cup \{\psi_2, \psi_1\} \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$

Combining them in the right way we will arrive at $\ulcorner X \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$. The second and third premiss are easily proven in $\mathrm{LK}_?$ by AX, DLA, DLC, and SW. – Suppose $V(\psi) = 1$ for all $\psi \in X\backslash\{\ulcorner\psi_1 \leftrightarrow \psi_2\urcorner\}$. Either $V(\psi_1) = V(\psi_2)$ or $V(\psi_1) \neq V(\psi_2)$. If the former, then $V(\ulcorner\psi_1 \leftrightarrow \psi_2\urcorner) = 1$. Thus, $V(\psi) = 1$ for all $\psi \in X$. Thus, there is $\psi \in \{\varphi_1, ..., \varphi_n\} \subseteq \{\varphi_1, ..., \varphi_n, \psi_1, \psi_2\}$ such that $V(\psi) = 1$. If the latter, then either $V(\psi_1) = 1$ or $V(\psi_2) = 1$. Thus, there is $\psi \in \{\varphi_1, ..., \varphi_n, \psi_1, \psi_2\}$ such that $V(\psi) = 1$. Thus, $X\backslash\{\ulcorner\psi_1 \leftrightarrow \psi_2\urcorner\} \Vdash\!\models_P \{\varphi_1, ..., \varphi_n, \psi_1, \psi_2\}$ (first premiss lemma). Now suppose $V(\psi) = 1$ for all $\psi \in (X\backslash\{\ulcorner\psi_1 \leftrightarrow \psi_2\urcorner\}) \cup \{\psi_2, \psi_1\}$. Then $V(\psi_1) = V(\psi_2)$ and therefore $V(\ulcorner\psi_1 \leftrightarrow \psi_2\urcorner) = 1$. Thus, $V(\psi) = 1$ for all $\psi \in X$. Thus, there is $\psi \in \{\varphi_1, ..., \varphi_n\}$ such that $V(\psi) = 1$. Thus, $(X\backslash\{\ulcorner\psi_1 \leftrightarrow \psi_2\urcorner\}) \cup \{\psi_2, \psi_1\} \Vdash\!\models_P \{\varphi_1, ..., \varphi_n\}$ (fourth premiss lemma). By induction assumption, we know from the two premiss lemmas that the first and the fourth premiss (see the list above) are $\mathrm{LK}_?$-provable. An application of CdIA to the first and second premiss and to the third and fourth premiss proves:

5. $\ulcorner(X\backslash\{\psi_1 \leftrightarrow \psi_2\}) \cup \{\psi_2 \to \psi_1\} \vdash ?\{\varphi_1, ..., \varphi_n, \psi_1, \varphi_1, ..., \varphi_n, \psi_1\}\urcorner$
6. $\ulcorner(X\backslash\{\psi_1 \leftrightarrow \psi_2\}) \cup \{\psi_2 \to \psi_1, \psi_2\} \vdash ?\{\varphi_1, ..., \varphi_n, \varphi_1, ..., \varphi_n\}\urcorner$

After shortening the consequents by SW and CT, we apply CdIA again proving:

7. $\ulcorner(X\backslash\{\psi_1 \leftrightarrow \psi_2\}) \cup \{\psi_1 \to \psi_2, \psi_2 \to \psi_1\} \vdash ?\{\varphi_1, ..., \varphi_n, \varphi_1, ..., \varphi_n\}\urcorner$.

Two applications of BIA and multiple applications of SW and CT then prove $\ulcorner X \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$.

(iii) Suppose $\varphi_k = \ulcorner\psi_1 \wedge \psi_2\urcorner \in \{\varphi_1, ..., \varphi_n\}$. Suppose $V(\psi) = 1$ for all $\psi \in X$. Then there is $\psi \in \{\varphi_1, ..., \varphi_n\}$ such that $V(\psi) = 1$. Let ψ_3 be so. Either $\psi_3 = \varphi_k = \ulcorner\psi_1 \wedge \psi_2\urcorner$ or $\psi_3 = \varphi_i$ (with $k \neq i$). If the former, then $V(\ulcorner\psi_1 \wedge \psi_2\urcorner) = V(\psi_1) = V(\psi_2) = 1$. Thus, there is $\psi \in \{\varphi_1, ..., \varphi_{k-1}, \varphi_{k+1}, ..., \varphi_n, \psi_1\}$ such that $V(\psi) = 1$ and there is $\psi \in \{\varphi_1, ..., \varphi_{k-1}, \varphi_{k+1}, ..., \varphi_n, \psi_2\}$ such that $V(\psi) = 1$. If the latter, then this follows directly. Thus, $X \Vdash_P \{\varphi_1, ..., \varphi_{k-1}, \varphi_{k+1}, ..., \varphi_n, \psi_1\}$ and $X \Vdash_P \{\varphi_1, ..., \varphi_{k-1}, \varphi_{k+1}, ..., \varphi_n, \psi_2\}$. By induction assumption, both $\ulcorner X \vdash ?\{\varphi_1, ..., \varphi_{k-1}, \varphi_{k+1}, ..., \varphi_n, \psi_1\}\urcorner$ and $\ulcorner X \vdash ?\{\varphi_1, ..., \varphi_{k-1}, \varphi_{k+1}, ..., \varphi_n, \psi_2\}\urcorner$ are $LK_?$-provable. CIC and $n-k$ applications of SW then prove $\ulcorner X \vdash ?\{\varphi_1, ..., \varphi_n\}\urcorner$. $\dashv$

$LK_?$ is only the auxiliary calculus in the obvious calculization. In order to arrive at a full calculus for propositional erotetic implication (PEI) one has to assemble multiple $LK_?$-proofs into what I will call a PEI-development[18]:

Definition 4.3 (PEI-development). –
$\mathfrak{P}$ *is a* PEI-*development of* Q_2 *from* Q_1 *relative to* X iff
(i) Q_1, Q_2 are $L_0^?$-questions and X is a set of formulas and
(ii) $\mathfrak{P}$ is a finite graph labeled with e*-sequents and all connected maximal subgraphs of $\mathfrak{P}$ are trees which are in accord with the rules AX, DLA, DLC, CT, SW, CIA1, CIA2, CIC, DIA, DIC1, DIC2, NIA, NIC, CdIA, CdIC, BIA1, BIA2, BIC and
(iii) for each $\varphi \in dQ_1$ there is a $Z \subseteq X$ and a connected maximal subgraph P of $\mathfrak{P}$ such that $\ulcorner Z \cup \{\varphi\} \vdash Q_2\urcorner$ is the root of P and
(iv) for each $\psi \in dQ_2$ there is a $Y \subset dQ_1$ and a $Z \subseteq X$ and a connected maximal subgraph P of $\mathfrak{P}$ such that $\ulcorner Z \cup \{\psi\} \vdash ?Y\urcorner$ is the root of P.

Here is an example PEI-development of '$?\{\mathsf{p}_1, \mathsf{r}\}$' from '$?\{\mathsf{p}, \mathsf{q}, \mathsf{r}\}$' relative to $\{$'$\mathsf{p} \vee \mathsf{q} \leftrightarrow \mathsf{p}_1$'$\}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{\mathsf{p}\} \vdash ?\{\mathsf{p}\}}\text{AX}}{\{\mathsf{p}\} \vdash ?\{\mathsf{p} \vee \mathsf{q}\}}\text{DIC1} \qquad \dfrac{}{\{\mathsf{p}_1\} \vdash ?\{\mathsf{p}_1\}}\text{AX}}{\{\mathsf{p} \vee \mathsf{q} \to \mathsf{p}_1, \mathsf{p}\} \vdash ?\{\mathsf{p}_1\}}\text{CdIA}}{\{\mathsf{p} \vee \mathsf{q} \leftrightarrow \mathsf{p}_1, \mathsf{p}\} \vdash ?\{\mathsf{p}_1\}}\text{BIA}$$

[18] The term 'development' is taken from Kneale and Kneale ([12, Sect. IX.3]) who diverge from the tree structure in a different fashion (downward branching) than we do here. For the graph-theoretical terminology see [8, Ch. 1].

$$\dfrac{}{\{\mathsf{p} \vee \mathsf{q} \leftrightarrow \mathsf{p}_1, \mathsf{p}\} \vdash ?\{\mathsf{p}_1, \mathsf{r}\}}\ \text{DLC}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\{\mathsf{q}\} \vdash ?\{\mathsf{q}\}}\ \text{AX}}{\{\mathsf{q}\} \vdash ?\{\mathsf{p} \vee \mathsf{q}\}}\ \text{DIC2} \qquad \dfrac{}{\{\mathsf{p}_1\} \vdash ?\{\mathsf{p}_1\}}\ \text{AX}}{\{\mathsf{p} \vee \mathsf{q} \to \mathsf{p}_1, \mathsf{q}\} \vdash ?\{\mathsf{p}_1\}}\ \text{CdIA}}{\{\mathsf{p} \vee \mathsf{q} \leftrightarrow \mathsf{p}_1, \mathsf{q}\} \vdash ?\{\mathsf{p}_1\}}\ \text{BIA}}{\{\mathsf{p} \vee \mathsf{q} \leftrightarrow \mathsf{p}_1, \mathsf{q}\} \vdash ?\{\mathsf{p}_1, \mathsf{r}\}}\ \text{DLC}$$

$$\dfrac{\dfrac{\dfrac{}{\{\mathsf{r}\} \vdash ?\{\mathsf{r}\}}\ \text{AX}}{\{\mathsf{r}\} \vdash ?\{\mathsf{r}, \mathsf{p}_1\}}\ \text{DLC}}{\{\mathsf{r}\} \vdash ?\{\mathsf{p}_1, \mathsf{r}\}}\ \text{SW}$$

$$\dfrac{\dfrac{\dfrac{}{\{\mathsf{p}_1\} \vdash ?\{\mathsf{p}_1\}}\ \text{AX} \qquad \dfrac{\dfrac{\dfrac{}{\{\mathsf{p}\} \vdash ?\{\mathsf{p}\}}\ \text{AX}}{\{\mathsf{p}\} \vdash ?\{\mathsf{p}, \mathsf{q}\}}\ \text{DLC} \qquad \dfrac{\dfrac{\dfrac{}{\{\mathsf{q}\} \vdash ?\{\mathsf{q}\}}\ \text{AX}}{\{\mathsf{q}\} \vdash ?\{\mathsf{q}, \mathsf{p}\}}\ \text{DLC}}{\{\mathsf{q}\} \vdash ?\{\mathsf{p}, \mathsf{q}\}}\ \text{SW}}{\{\mathsf{p} \vee \mathsf{q}\} \vdash ?\{\mathsf{p}, \mathsf{q}\}}\ \text{DIA}}{\{\mathsf{p}_1 \to \mathsf{p} \vee \mathsf{q}, \mathsf{p}_1\} \vdash ?\{\mathsf{p}, \mathsf{q}\}}\ \text{CdIA}}{\{\mathsf{p} \vee \mathsf{q} \leftrightarrow \mathsf{p}_1, \mathsf{p}_1\} \vdash ?\{\mathsf{p}, \mathsf{q}\}}\ \text{BIA}$$

$$\dfrac{}{\{\mathsf{r}\} \vdash ?\{\mathsf{r}\}}\ \text{AX}$$

Note that this is not five, but *one* PEI-development. The upper three connected maximal subgraphs substantiate condition (iii) of the definiens of a PEI-development and correspond to the first condition of the definiens of erotetic implication. The lower two connected maximal subgraphs substantiate condition (iv) of the definiens of a PEI-development and correspond to the second condition of the definiens of erotetic implication. All five subgraphs are proofs of the auxiliary calculus $\mathrm{LK}_?$.

Now we can define '$\vdash_{PEI}$' as a ternary metalanguage predicate (not as a sequent-forming operator):

Definition 4.4 (Provability by PEI-development). –
$Q_1, X \vdash_{PEI} Q_2$ iff
there is a PEI-development $\mathfrak{P}$ of Q_2 from Q_1 relative to X.

This yields the following adequacy result:

Theorem 4.5 (PEI-adequacy w.r.t. propositional erotetic implication). –
For all sets X of $L_0^?$-formulas and all $L_0^?$-questions Q_1 and Q_2:
$Q_1, X \vdash_{PEI} Q_2$ *iff* $Im(Q_1, X, Q_2)$.

Proof. The left-to-right direction follows directly from the definition of PEI-developments and the finite $\mathrm{LK}_?$-adequacy w.r.t. mc-entailment. The right-to-left direction has only one minor difficulty: the possible infinity of X. However, compactness holds for classical mc-entailment ([22, p. 40]). Now, if $Im(Q_1, X, Q_2)$, then, by definition of erotetic implication, (i) for all $\varphi \in dQ_1$: $X \cup \{\varphi\} \Vdash dQ_2$ and (ii) for all $\psi \in dQ_2$ there is $Y \subset dQ_1, Y \neq \emptyset$ such that: $X \cup \{\psi\} \Vdash Y$. Each of these entailments works with a finite subset, say X_i of

X instead of X. For these the finite $\mathrm{LK}_?$-adequacy w.r.t. mc-entailment can be applied, providing the needed $\mathrm{LK}_?$-proofs for the PEI-development witnessing $Q_1, X \vdash_{PEI} Q_2$. ⊣

§5. Critique of the Obvious Approach and First-Order Erotetic Implication. The PEI-calculus can be easily criticized for diverging from a number of conventions.[19] One convention is the tree-structure of proofs; PEI-developments are not trees.[20] However, the tree-structure is usually viewed as a good format for a calculus relative to a "reading" or an "interpretation" of this format and the sequents in it. Rejecting PEI-developments because of the non-tree format presupposes that said reading cannot be acceptibly extended to sets of trees. It is not quite clear why this should be so. Alternatively, one may invoke the naturalness of trees, but, again, it is unclear whether the tree-structure or the development-structure is particularly natural. "Naturalness" can only be assessed relative to a certain "interpretation" of the format of a caluclus. If, however, a calculus is intended as what the name says, i.e. an apparatus for calculating something, then developments and trees are on a par, as long as they do this job *adequately*. As we have seen above, adequacy *is* granted.

Another rejected convention, this time an IEL-specific convention, is manifest in the inclusion of single-answer questions, zero-answer questions, and questions with multiple occurrences of the same answer(s). Note that the questions of which and from which some graph is a PEI-development are still $\mathrm{L}_0^?$-questions and thus have at least two answers and no answer occurs twice. So the unconventionality concerns only intermediary "questions", including, for example, the succedent of the only e*-sequent in the last $\mathrm{LK}_?$-proof of the PEI-development example in the last section ('$\{\mathsf{r}\} \vdash ?\{\mathsf{r}\}$'). Again, if PEI is seen as a mere device for calculation, this should be no problem. But if one wants a "natural" question in the succedent of each node in the development, then the occurrence of such "non-standard" questions is a problem.

The assessment of PEI seems to depend on its "reading" and on what programmatic role one ascribes to it: Is it a pure calculating device within a certain domain of $\mathrm{L}_0^?$-expressions or is it a model and guide for practices of posing questions in informal and semi-formal contexts? In the first chapter of [24] Wiśniewski dedicates more than 20 pages to the study of paradigmatic natural-language settings in which questions arise. This suggests that IEL and IEL-calculi should indeed be seen as closely related to practices of posing questions. From this perspective neither the tree-metacalculus $\mathrm{PMC_E}$ nor the development-metacalculus PEI are good calculizations – at least not on their own: It is not obvious in what way they model real practices of posing questions. However, assuming

[19] As for the merits of the calculus, a commentator drew my attention to the fact, that failed PEI-derivations tell the user, in a sense, why an implication might not hold, i.e. which clause of the definition might not be satisfied. It is just a matter of what kind of subgraph is missing, one that is supposed to be there according to clause (iii) of the PEI-development definition or one that is supposed to be there according to clause (iv). – The ups and downs of the PEI-calculus regarding automatization have not been considered for the paper.

[20] Of course, it is always possible to amend the PEI-calculus by "artificially" connecting all otherwise mutually disconnected subgraphs (the $\mathrm{LK}_?$-proofs) which make up each development. In this amended calculus each development can be a tree again.

that Wiśniewski was after the practices of posing questions, then it seems odd that he treats rules of posing questions only very marginally in a subsection (8.5.3) of the last chapter of [24].[21] Wiśniewski's more recent book [25] is more frugal when investigating natural-language questions in settings that resemble evocation or erotetic implication (about seven pages in Ch. 5). Notably, one chapter of the book concerns Socratic transformations and five chapters erotetic search scenarios (e-scenarios). In both tools erotetic implication plays an important role; in the case of Socratic transformations, metatheorems about erotetic implication are converted into rules of stepping from one question to another. Anyway, to what extent the role of erotetic implication in e-scenarios is reconcilable with the calculization of erotetic implication (or lack thereof), is quite unclear.

Anyway, as it is now, there is no caclulus for evoking or implying questions, in the sense of IEL, which is modelled directly upon natural practices of asking questions.[22] However, we can *calculize* propositional evocation and propositional erotetic implication with $\mathrm{PMC_E}$ and PEI, respectively. This is a partial success. Can the "obvious approach" to propositional erotetic implication be extended to include first-order erotetic implication and furthermore to evocation? There seems to be no prima facie reason to suspect that it cannot be extended to include first-order frameworks. Gentzen's calculus LK includes quantifier rules which can be directly translated in analogy to the propositional rules above, resulting in a calculus which may be dubbed the FOEI-calculus (first-order erotetic implication).

$$\text{UIA:}\frac{X \cup \{\psi(\alpha/\xi)\} \vdash ?\{\varphi_1, ..., \varphi_n\}}{X \cup \{\bigwedge \xi\psi\} \vdash ?\{\varphi_1, ..., \varphi_n\}} \qquad \text{EIC:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi(\alpha/\xi)\}}{X \vdash ?\{\varphi_1, ..., \varphi_n, \bigvee \xi\psi\}}$$

$$\text{UIC:}\frac{X \vdash ?\{\varphi_1, ..., \varphi_n, \psi\}}{X \vdash ?\{\varphi_1, ..., \varphi_n, \bigwedge \xi\psi\}} \text{ provided } \xi \text{ does not occur in } \varphi_1, ..., \varphi_n \text{ or any element of } X$$

$$\text{EIA:}\frac{X \cup \{\psi\} \vdash ?\{\varphi_1, ..., \varphi_n\}}{X \cup \{\bigvee \xi\psi\} \vdash ?\{\varphi_1, ..., \varphi_n\}} \text{ provided } \xi \text{ does not occur in } \varphi_1, ..., \varphi_n \text{ or any element of } X$$

[21] Arguably, the concept of rules employed in [24] has less to do with regulating acts of posing questions than one might think: According to sect. 8.5.3, *any* set of ordered pairs (X, φ) such that X evokes φ is an e_1-rule. Hence, in particular, $\mathbf{R} = \{(X, \varphi) \mid X \text{ evokes } \varphi\}$ is an e_1-rule (and all other e_1-rules are subsets of $\mathbf{R}$). Since, as will be shown below (Sect. 6), classical first-order evocation is not calculizable, there is no *general* way to determine whether some X and some φ together satisfy the e_1-rule $\mathbf{R}$. Thus, it is not *generally* the case, that a language user can decide whether her inference from some X to some φ is in accord with a given e_1-rule for classical first-order evocation. It is a different story with *syntactically specified* e_1-rules, but with regard to the generality of the definition of 'e_1-rule' in [24], it appears, rules lose their regulativity, i. e. their action-guiding character, in IEL.

[22] Even Millson's proposal, which comes with an interpretation of its sequents [19, Sect. 2], does not map *practices* of asking questions but *entitlements* to do so and how one *infers* that such entitlements obtain. Thus, in a sense, it is not an erotetic calculus but an assertoric calculus whose subject matter are erotetic (and assertoric) entitlements. – Meheus claims the proposal in [18, p. 136] to "explicate actual reasoning processes". Whether this is true is debatable, but the option of withdrawing questions makes the proposal appear as quite natural in some respects.

However, in first-order IEL there is usually *at least* one new kind of question, namely those of the form $\ulcorner ?S(\varphi)\urcorner$ whose set of direct answers includes all formulas resulting from φ by substitution of individual constants for the free variables. If the concept of an $\mathrm{L}^?_1$-question of the first kind ($\ulcorner ?\{\varphi_1, ..., \varphi_n\}\urcorner$) included infinite sequences of φ_i, then $\mathrm{L}^?_1$-questions of the second kind ($\ulcorner ?S(\varphi)\urcorner$) could be reduced to $\ulcorner ?\{\varphi(\alpha_1/\xi), \varphi(\alpha_2/\xi), ...\}\urcorner$, where $\alpha_1, \alpha_2, ...$ is a complete (but infinite) list of the individual constants.[23]

However, this reduction alone does not allow for the application of the "obvious approach" to the first-order case. As the rules are now (including UIA, UIC, EIA, EIC), none enables an inference to a sequent which has an infinite question ($\ulcorner ?\{\varphi(\alpha_1/\xi), \varphi(\alpha_2/\xi), ...\}\urcorner$) in its succedent. This can be remedied in a number of ways (e.g. liberalizing DLC to allow for "dilution to infinity"), but there are further problems. Note that even in finite domains and with finitely many individual constants in the language the disjunction of all substitution instances of a formula φ is not equivalent to $\ulcorner \bigvee \xi_1 ... \bigvee \xi_n\ \varphi\urcorner$ (with $\xi_1, ..., \xi_n$ being all the free variables of φ). To put it realistically: Some entities may lack a name referring to them and these nameless entities may well be the only witnesses for an existential formula. This is why Wiśniewski limits his interpretations, which determine mc-entailment, to be "normal" interpretations in which each element of the domain has a name in the interpreted language ([24, p. 105, 122], [25, p. 32]). With respect to any of these interpretations, if it makes true $\ulcorner \bigvee \xi_1 ... \bigvee \xi_n\ \varphi\urcorner$, then it makes true at least one of the substitution instances of φ (and the converse).[24]

This illustrates how the limitation to normal interpretations changes the mc-entailment relation. While usually $\{\ulcorner \bigvee \xi\varphi\urcorner\} \not\Vdash \{\psi \mid \text{there is a constant } \alpha : \psi = \varphi(\alpha/\xi)\}$, if limited to normal interpretations we have $\{\ulcorner \bigvee \xi\varphi\urcorner\} \Vdash \{\psi \mid \text{there is a constant } \alpha : \psi = \varphi(\alpha/\xi)\}$. Effectively, Wiśniewski presupposes substitutional quantification and thereby a different logic, which poses considerable calculization issues due to its infinitary character. – In summary, there are a number of unsolved problems in applying the approach of the preceding section to first-order erotetic implication. Here is a sketchy list:

- Should first-order erotetic implication be construed with a standard first-order mc-entailment relation or with its variant resulting from the employment of substitution semantics instead of model theory?
- How can the FOEI-calculus be modified to allow for inferences to sequents in which the succedent is either an infinite question ($\ulcorner ?\{\varphi_1, \varphi_2, ...\}\urcorner$) or an S-question ($\ulcorner ?S(\varphi)\urcorner$)?

[23] This is the simplest case where φ has only one free variable (ξ).

[24] There may still be no (object-language) disjunction of substitution instances of φ which is equivalent to $\ulcorner \bigvee \xi_1 ... \bigvee \xi_n\ \varphi\urcorner$ if the universe (and the set of individual constants) is allowed to be infinite again. Whether this is a problem or not, depends on whether one wants to admit infinitary elements (cf. [2]). – The whole problem relates to ω-incompleteness, which was explicitly recognized by Meheus ([18, p. 141]).

- Can other kinds of first-order questions, which are envisaged in IEL, figure in an erotetic implication calculus?[25]

Hence, the problem of calculizing first-order erotetic implication remains unsolved, although some ideas are available.

§6. The Obvious Approach Applied to Evocation and Uncalculizability. How does the approach from Section 4 translate to evocation? The answer is simple, regardless of whether one refers to propositional or first-order evocation: The "obvious approach" does *not* apply to evocation, because evocation involves a *negative* entailment clause in its definition:

$E(X, Q)$ iff $X \Vvdash dQ$ and there is no $\varphi \in dQ$ such that $X \models \varphi$.

The "obvious approach" turned the two clauses about positive entailments in the definition of erotetic implication into the clauses (iii) and (iv) of the definition of PEI-development while exploiting a suitable mc-calculus. However, the second conjunct in the definiens for evocation requires proof that certain formulas do *not* follow. This means, a standard calculus adequate with respect to propositional entailment, be it an sc-calculus or an mc-calculus, is not enough. One needs a calculus which is adequate with respect to non-entailment. If one has such a calculus, like G3cp from [20, Ch. 3], then the approach from the preceding section can be applied to evocation. Wiśniewski's calculus $\mathrm{PMC_E}$ does not start with such a negative calculus, but, as stated in fn. 16 above, his calculus effectively includes a negative calculus for classical propositional logic.[26]

With first-order evocation there are problems similar to first-order erotetic implication. However, even for finite questions of the first kind ($\ulcorner ?\{\varphi_1, ..., \varphi_n\}\urcorner$) there are decidability problems with calculizing evocation.[27] In fact, one can prove:

Theorem 6.1 (Uncalculizability of First-Order Evocation). *There is no calculus defining $\vdash_{FOE}$ such that for all sets X of $L_1^?$-formulas and all $L_1^?$-choice-questions:*

$E(X, Q)$ iff $X \vdash_{FOE} Q$.

Proof. Suppose there was a calculus (in the sense of sect. 1) such that for all sets of formulas X and all questions of the form $\ulcorner ?\{\varphi_1, ..., \varphi_n\}\urcorner$ we have $X \vdash_{FOE} \ulcorner ?\{\varphi_1, ..., \varphi_n\}\urcorner$ iff $E(X, \ulcorner ?\{\varphi_1, ..., \varphi_n\}\urcorner)$. To count as a calculus in the usual sense the number of its proofs should be denumerable. If all proofs are finite arrangements of expressions of $\mathrm{L}_1^?$, as is presupposed, then this denumerability is guaranteed. We will show that $X \not\models \varphi$ is equivalent to a certain FOE-consequence $X' \vdash_{FOE} Q$ where one direct answer to Q is a proxy for φ.

[25] Apart from the relatively simple S-questions, there are at least O-, T-, U-, and W-questions in IEL, corresponding to questions with more sophisticated types of answers (conjunctive answers, answers with completeness claims, etc.). See [24, Sect. 3.1].

[26] By speaking of the *defeasible character* of evocation Millson, too, discusses to what extent the second clause of the definition of propositional evocation is recognizable in Wiśniewski's calculus and in his own ([19]).

[27] That there are problems with evocation resulting from the undecidability of first-order logic was already seen by Meheus ([18, p. 139–140]). However, these problems are only very briefly mentioned by Meheus.

For the purpose of the proof we take $\Phi_1, \Phi_2, ...$ to be a list of all unary predicates. Furthermore we have a function *inc* which maps any formula φ to a formula $inc(\varphi)$ which is just like φ except that each unary predicate Φ_i is replaced by Φ_{i+1}. For sets X of formulas we declare: $inc(X) = \{\varphi \mid \text{there is } \psi \in X : inc(\psi) = \varphi\}$. We can observe: $X \models \varphi$ iff $inc(X) \models inc(\varphi)$.[28]

Now let us pick some abitrary set of formulas X and some arbitrary formula φ. In $inc(X \cup \{\varphi\})$ the predicate Φ_1 does not occur. Let α be any individual constant of $\mathrm{L}_1^?$. As a special case of the adequacy of the hypothesized FOE-calculus with respect to evocation we have (cf. fn. 16):

$inc(X) \vdash_{FOE} \ulcorner ?\{inc(\varphi), \Phi_1(\alpha), \neg\Phi_1(\alpha)\}\urcorner$
iff $E(inc(X), \ulcorner ?\{inc(\varphi), \Phi_1(\alpha), \neg\Phi_1(\alpha)\}\urcorner)$

Note that the question at issue here ($\ulcorner ?\{inc(\varphi), \Phi_1(\alpha), \neg\Phi_1(\alpha)\}\urcorner$) is what in IEL is called a *safe* question. That is, its set of answers is mc-entailed by any set. This is obvious because the question contains a closed formula and its negation as two answers. Thus, when applying the definition of evocation we can disregard the first conjunct in the definiens as trivially satisfied. We can reformulate the iff-sentence above thus:

$inc(X) \vdash_{FOE} \ulcorner ?\{inc(\varphi), \Phi_1(\alpha), \neg\Phi_1(\alpha)\}\urcorner$
iff there is no $\psi \in \{inc(\varphi), \ulcorner\Phi_1(\alpha)\urcorner, \ulcorner\neg\Phi_1(\alpha)\urcorner\}$ such that $inc(X) \models \psi$
iff $inc(X) \not\models inc(\varphi)$ and $inc(X) \not\models \ulcorner\Phi_1(\alpha)\urcorner$ and $inc(X) \not\models \ulcorner\neg\Phi_1(\alpha)\urcorner$

Now we can exploit the use of the *inc* function: Since Φ_1 does not occur in $inc(X)$, each of the latter two conjuncts is equivalent to the consistency of $inc(X)$. This consistency is already implied by the first conjunct. Hence:

$inc(X) \vdash_{FOE} \ulcorner ?\{inc(\varphi), \Phi_1(\alpha), \neg\Phi_1(\alpha)\}\urcorner$ iff $inc(X) \not\models inc(\varphi)$

Recalling that $X \models \varphi$ iff $inc(X) \models inc(\varphi)$ for all X, φ, we can now say:

$inc(X) \vdash_{FOE} \ulcorner ?\{inc(\varphi), \Phi_1(\alpha), \neg\Phi_1(\alpha)\}\urcorner$ iff $X \not\models \varphi$

This holds for all sets X of formulas and all formulas φ. Now we can start setting up a decision algorithm for first-order entailments, which contradicts Church's undecidability of first-order logic ([5]). First enumerate all FOE-proofs $\mathfrak{P}_1, \mathfrak{P}_2, ...$ and enumerate all proofs $\mathfrak{H}_1, \mathfrak{H}_2, ...$ of a standard Hilbert first-order calculus. In order to check whether $X \models \varphi$ for any set X of formulas and any formula φ, go through the merged enumeration $\mathfrak{P}_1, \mathfrak{H}_1, \mathfrak{P}_2, \mathfrak{H}_2, ...$ Eventually, either an FOE-proof supporting $inc(X) \vdash_{FOE} \ulcorner ?\{inc(\varphi), \Phi_1(\alpha_1), \neg\Phi_1(\alpha_1)\}\urcorner$, equivalent to $X \not\models \varphi$, or a Hilbert-proof supporting $X \models \varphi$ has to occur. $\dashv$

[28] Suppose $X \models \varphi$. Suppose $inc(X)$ is true in the model $M = \langle D, I\rangle$. Define $M' = \langle D, I'\rangle$ so that it is just like M except for the interpretations of the unary predicates: $I'(\Phi_i) = I(\Phi_{i+1})$. Then X is true in M'. By assumption, φ is true in M'. Then $inc(\varphi)$ is true in M. Thus $inc(X) \models inc(\varphi)$. The right-to-left direction is analogous.

From the last lines of the proof two things should be obvious: (i) The theorem does not hold for any decidable fragment of first-order logic or other decidable logics. (ii) The theorem *does* generalize for any undecidable logic, including undecidable propositional logics like some relevance logics (cf. [17]).

§7. A Programmatic Summary. The table from Section 1 can be updated:

	propositional	**first-order**
Evocation	[26], [19]	unsolvable (Sect. 6)
Erotetic Implication	solved (Sect. 4; cf. [19])	unsolved (but some ideas in Sect. 5)

The result in the upper right corner of the table is an opportunity to take a step back and ask: Is this just a technical result or does it have implications for the way one should look at IEL? When developing a grasp of the arising of questions and how it yields the formal concepts of evocation and erotetic implication, Wiśniewski frames his enterprise as an explicative one ([24, p. 12-13, 26]). He echoes Rudolf Carnap ([4, p. 5]) when he remarks that "in the case of explication the correspondence between the "old" and "new" concepts need not be a complete coincidence."

The discussion on Carnapian explication is extensive (cf. [7]) and the issue as to how similar explicandum and explicatum have to be is controversial, but in the case at hand the discrepancy is quite considerable: Should evocation still count as a successful explicatum of the arising of questions from sets of declarative closed formulas if (first-order) evocation cannot be regulated by any calculus while the regulatability of the arising of natural language questions is at least an open issue? Wiśniewski's explication *is* successful with respect to his own measure of explicative adequacy, which consists in four conditions ([24, p. 12]): If X evokes Q, then (1) no direct answer to Q belongs to X, (2) no direct answer to Q is entailed by X, (3) if all the formulas in X are true, the question Q must have a true direct answer, (4) each presupposition of Q is entailed by X. However, this measure includes at least two conditions not clearly related to the processes of question arising (conditions 2 and 3). Since entailment is a rather theory-laden formal relation, it is unplausible that we assess our usual practices of asking questions against it; although, we may be using some other somewhat similar relation for such an assessment. Furthermore, against condition 3, we often feel a question is raised by some X despite our ignorance about whether there is a true direct answer to the question. This may be related to the fact, that we often ask questions without being aware of the set of direct answers that this question is associated with.[29] – Note that one of the two problematic conditions, namely condition 2, prevents, if satisfied, first-order evocation from being governed by a definitive set of rules. If one rejects Wiśniewski's measure of adequacy, then it is not clear according to which alternative measure a given concept of evocation successfully explicates the arising of questions from sets of formulas.

[29] From this consideration one could try to argue against the set-of-answers methodology of IEL. In this paper, I do not want to present such a kind of critique.

The larger problem one may suspect beneath this can be described as follows: IEL's core relations, i.e. evocation and erotetic implication, are defined by "semantic" vocabulary. They are not defined by the stepping from certain object-language premisses to object-language conclusions, be they questions or formulas. Alternatively one may proceed thus: Once questions are represented in a formal language (as IEL allows), one can formally represent the consecutive stepping from formulas/questions to formulas/questions as a *sequence* of object-language expressions. One can then try to find *rules* which describe a selection of such sequences, or rather: The setting of rules of sequentially stepping from expressions to expressions establishes a formal practice of arriving at formulas or at questions. With respect to natural-language practices, this setting of formal rules together with the underlying schema of formalization may *support* the natural practices of asking questions. In addition, if the imposed rules satisfy certain basic requirements, there is no risk of any uncalculizability.

The analogy with declarative logic is glaring:[30] Setting inference rules establishes a formal practice of arriving at conclusions and supports the informal counterpart; it rectifies "fallacies" and stabilizes "good" inferences. Once such rules are set, one can abstract from specific rule-compliant sequences of formulas (derivations) and talk about the scope of action spanned by the practice (derivability). This *scope concept* ([21, Sect. 4]) of derivability is determined by the rules and may or may not be equivalent to a concept like semantic entailment, which is established differently and is not a scope concept. Equally, any set of rules describing or prescribing a practice of raising questions spans a scope of action. The resulting scope concepts can be called evocation[31] or erotetic implication and they may or may not be equivalent to certain other concepts defined in a different way. (There surely are benefits if certain equivalences obtain!)

Admittedly, the practice of stepping from one question to another is treated in IEL, namely in the framework of Socratic transformations. However, it is not the case that this stepping from one question to another is considered primitive or at least "semantically uninformed". Quite to the contrary, the "semantic" relation of erotetic implication is already presupposed for the setting up of a calculus of Socratic transformations:

> "Statements (8.11) and (8.19) specified above are examples of metatheorems stating what questions are [erotetically] implied by what questions. Metatheorems of this kind enable us to formulate *erotetic rules*, being rules of transitions from questions to questions. A set of rules of this kind together with a characterization of *basic sequents* [...] constitute an *erotetic calculus*." [25, p. 94]

Here the suggested order of things is quite evident: *First* the IEL logician needs metatheorems of erotetic implication (whose connection to practices of asking

[30]Similar analogies have been recognized before, e.g.: "Just as deductive reasoning serves as a normative yardstick in the field of reasoning with declaratives [...] so is erotetic implication a normative yardstick for reasoning with questions [...]" ([15, p. 91]).

[31]The term 'evokability' would be more suitable for the scope concept, because at least etymologically 'evocation' is to be understood as referring to some kind of utterance – and has no morpheme denoting the potentiality befitting a scope concept. Compare 'derivation'/'derivability', 'proof'/'provability', 'computation'/'computability', etc..

question has been problematized above). *Second* these metatheorems are converted into rules, regulating and setting up a practice of asking questions.

In sum, there are two kinds of research programmes: (i) One programme presupposes "semantic" relations and either tries to axiomatize them, effectively providing "syntactic" calculi, or converts such a "semantic" relation into rules, again effectively providing a calculus. (If this is done as with the Socratic transformations, then the resulting calculus is a "syntactic" one, although "semantically informed"; in other cases there may be no reason to consider the resulting calculus "syntactic" due to the presupposed "semantic" relation.) Those adhering to this programme, if they are interested in the actual practices of raising questions, need to provide an account of what the calculi have to do with such practices of raising questions when both realms are connected only indirectly and not quite transparently via a "semantic" apparatus.[32] (ii) The other programme starts with these actual practices and, after formalization (which is not problematized here; cf. [25, ch. 1 and Sect. 2.2-2.3]), puts the practice under rules. These rules are not subject to "semantic" justification and thus do not presuppose "semantic" relations. The resulting scope concept allows one to express what is *in principle* evokable/askable/raisable if one stays within the confines of the practice as regulated by the rules provided. Acknowledging the adjustment of the informal practice through formalization and through the selection of specific rules, there is at least an intuitive understanding of what the formal scope concept, which may be named 'evocation' or 'erotetic implication', has to do with the initial practice.

The IEL research that I considered for this paper is largely concerned with the first programme.[33] Possibly, Meheus's [18] is an exception. At any rate, I propose to (more decidedly) follow the second programme within the IEL paradigm.[34] The current concepts of evocation and erotetic implication can serve as a point of orientation. In particular, the results concerning calculization of IEL suggest a number of questions subsumable under the second programme: (a) Since propositional erotetic implication is calculizable, what calculizations reflect our natural practices of proceeding from an implying question to an implied question via sequentially ordered intermediary steps? Do we have to rely on semantic intuitions? (b) How can we generalize such "natural" calculi to include first-order erotetic implication? (c) Since we now know that the calculization of

[32][16] may serve as a starting point for investigating the relation between IEL and natural language questions, although the issues I try to point to are not solved there since the role of the mediating formal semantics is not clear.

[33]This holds for Millson's calculus presented in [19], too, which is not concerned with stepping from (questions and) assertions to questions but with stepping from conditional entitlements of making assertions and asking questions to other such entitlements. In a sense, Inferential Erotetic Logic, when concerned with evocation or erotetic implication, is not inferential in that it is not *based* on inferential steps from expressions to expressions. Even some alternative approaches to erotetic implication (cf. [15, Sect. 3.2], [25, p. 71-72 and Sect. 7.4]) do not frame this relation as inferential in any obvious sense.

[34]A historical note: It seems that in the past, e.g. in classical first-order logic and modal logic, scholars tended to follow the second programme before becoming involved with "semantics" and thus enabling the first programme. However, the dawn of model theory and of possible worlds semantics undoubtedly enriched or, in the case of modal logic, jump-started the respective enterprise.

first-order evocation is not attainable, how close can we get to first-order evocation if we follow the second programme starting with the natural language scenarios that IEL scholars take to be typical first-order evocation scenarios? Presupposing a solution to this issue, we may further ask: (d) Does the adaptation of this solution to the propositional case yield propositional evocation as defined and axiomatized by Wiśniewski?

REFERENCES

[1] Ignacio Angelelli, *The techniques of disputation in the history of logic*, ***The Journal of Philosophy***, vol. 67 (1970), no. 20, pp. 800–815.

[2] John L. Bell, *Infinitary logic*, ***The Stanford Encyclopedia of Philosophy*** (Edward N. Zalta, editor), Metaphysics Research Lab, Stanford University, winter 2016 ed., 2016, URL: https://plato.stanford.edu/archives/win2016/entries/logic-infinitary/.

[3] Nuel D. Belnap and Thomas B. Steel, ***The logic of questions and answers***, Yale University Press, New Haven/London, 1976.

[4] Rudolf Carnap, ***Logical foundations of probability***, University of Chicago Press, Chicago, 1950.

[5] Alonzo Church, *A note on the Entscheidungsproblem*, ***The Journal of Symbolic Logic***, vol. 1 (1936), no. 1, pp. 40–41.

[6] Ivano Ciardelli, Jeroen Groenendijk, and Floris Roelofsen, ***Inquisitive Semantics***, OUP, Oxford, 2019.

[7] Moritz Cordes and Geo Siegwart, *Explication*, ***Internet Encyclopedia of Philosophy*** (James Fieser and Bradley Dowden, editors), 2018, URL: https://www.iep.utm.edu/explicat/.

[8] Reinhard Diestel, ***Graph theory***, Springer, Berlin, 2006.

[9] Gerhard Gentzen, *Untersuchungen über das logische Schließen I*, ***Mathematische Zeitschrift***, vol. 39 (1935), no. 2, pp. 176–210.

[10] David Harrah, *The logic of questions*, ***Handbook of philosophical logic*** (Dov M. Gabbay and Franz Guenthner, editors), vol. 8, Springer, 2nd ed., 2002, pp. 1–60.

[11] Jaakko Hintikka, ***Inquiry as inquiry: A logic of scientific discovery***, Kluwer, Dordrecht/Boston/London, 1999.

[12] William Kneale and Martha Kneale, ***The development of logic***, Clarendon Press, Oxford, 1962.

[13] Tadeusz Kubiński, ***An outline of the logical theory of questions***, Akademie-Verlag, Berlin, 1980.

[14] Johann Heinrich Lambert, ***Neues Organon oder Gedanken über die Erforschung und Bezeichnung des Wahren und dessen Unterscheidung vom Irrthum und Schein***, Johann Wendler, Leipzig, 1764.

[15] Dorota Leszczyńska-Jasion, ***From questions to proofs: Between the logic of questions and proof theory***, Faculty of Social Science Publishers Adam Mickiewicz University in Poznań, Poznań, 2018.

[16] Paweł Łupkowski, ***Logic of questions in the wild. Inferential Erotetic Logic in information seeking dialogue modelling***, College Publications, 2015.

[17] Edwin Mares, *Relevance logic*, ***The Stanford Encyclopedia of Philosophy*** (Edward N. Zalta, editor), Metaphysics Research Lab, Stanford University, spring 2014 ed., 2014, URL: https://plato.stanford.edu/archives/spr2014/entries/logic-relevance/.

[18] Joke Meheus, *Adaptive logics for question evocation*, ***Logique & Analyse***, vol. 173-174-175 (2001), pp. 135–164.

[19] Jared Millson, *A cut-free sequent calculus for defeasible erotetic inferences*, ***Studia Logica***, vol. 107 (2019), pp. 1279–1312, URL: https://doi.org/10.1007/s11225-018-9839-z.

[20] Sara Negri and Jan von Plato, ***Structural proof theory***, CUP, Cambridge, 2001.

[21] Friedrich Reinmuth and Geo Siegwart, *Inferential acts and inferential rules: The intrinsic normativity of logic*, ***Analyse und Kritik***, vol. 38 (2016), no. 2, pp. 417–431.

[22] D. J. Shoesmith and T. J. Smiley, ***Multiple-conclusion logic***, CUP, London/New York/Melbourne, 1978.

[23] Tomasz Skura and Andrzej Wiśniewski, *A system for proper multiple-conclusion entailment*, ***Logic and Logical Philosophy***, vol. 24 (2015), pp. 241–253.

[24] Andrzej Wiśniewski, ***The posing of questions. logical foundations of erotetic inferences***, Kluwer, Dordrecht/Boston/London, 1995.

[25] ———, ***Questions, inferences, and scenarios***, College Publications, Milton Keynes, 2013.

[26] ———, *An axiomatic account of question evocation: The propositional case*, ***Axioms***, vol. 5 (2016), no. 2, URL: https://doi.org/10.3390/axioms5020014.

[27] ———, *Deduction and reduction theorems for Inferential Erotetic Logic*, ***Studia Logica***, vol. 106 (2018), no. 2, pp. 295–309.

DEPARTMENT OF PHILOSOPHY
UNIVERSITY OF GREIFSWALD
GERMANY
E-mail: cordesm@uni-greifswald.de