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General Proof Theory of Classical Propositional Logic: One Size Fits All

Allard M. Tamminga*

Abstract

In this paper both a general Gentzen-style system of natural deduction and a general Gentzen-style sequent calculus will be presented, with which—according to the interpretation of the variables for logical symbols—exactly all tautologies, satisfiable formulas, rejectable formulas, and contradictions of classical propositional logic can be derived. In this way, it is shown that systems deriving these classes of formulas do not need alternative proof-structures, as supposed in all the literature on theories of rejection.

1 Theories of Rejection

Jan Łukasiewicz was the first to introduce the concept of 'rejection' into formal logic. In the 1921 paper 'Two-valued Logic'[1] Łukasiewicz followed Brentano in adding to Frege's idea of assertion Brentano's idea of rejection. In his early works, Łukasiewicz argued that a proposition must be rejected if and only if it is false, parallel with Frege's condition for the assertion of a proposition. Later on, starting with *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Łukasiewicz redefined the concept of rejection to cover not only false propositions, but also propositions which are false under at least one interpretation.[2] Furthermore, he introduced syntactical techniques to *derive* all rejectable, *i.e.*, non-tautological, statements. By using the symbol '$\vdash$' for assertion (indicating theoremhood) and '$\nvdash$' for rejection (indicating non-theoremhood), what Łukasiewicz added to *classical propositional logic (CPL)* is the following:

Axiom $\nvdash p$, where p is a fixed propositional variable
Detachment If $\vdash \phi \rightarrow \psi$ and $\nvdash \psi$, then $\nvdash \phi$
Substitution If $\nvdash \psi$ and ψ can be obtained from ϕ by substitution, then $\nvdash \phi$.

This system is first described in Łukasiewicz [5], where Łukasiewicz also propounded a complete system of rejection for Aristotle's syllogistics, after some technical problems had been solved by Jerzy Słupecki. Łukasiewicz also tried to construct systems of rejection for the *intuitionistic propositional logic (IPL)* and for his own version of modal logic.

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[1]Cf. Łukasiewicz [4].
[2]Hence, it would be more appropriate to speak of theories of *rejectability*, which includes a modal component, instead of theories of *rejection*, but I will stick to the traditional nomenclature, coined by Łukasiewicz.

Until recently, all proposed rejection systems suffer badly from lack of elegance, because of some awkward, strongly semantical motivated derivation rule.[3] For classical propositional logic more elegant rejection systems have been constructed by Goranko [1] and Tamminga [7]. Still, the proof-structures of these systems deviate from the standard ones.

In this article I make the following claim: In classical propositional logic, no alternative proof-structure is needed for the construction of syntactical systems characterizing the satisfiable, the rejectable, or the contradictory formulas. I shall establish this claim by constructing two instances of it—the general Gentzen-style system of natural deduction **ND** and the general Gentzen-style sequent calculus **SC**—and by proving their equipollence, correctness, and completeness. Although this paper has been motivated primarily by the study of theories of rejection, in the proofs all the attention is devoted to theories of satisfiability (the proofs for theories of rejection are, surprisingly, completely analogous). This procedure is adopted for the sake of readability.

2 Language and Semantics

In order to establish the claim made in the preceding section, we need a generalization of the concept of classical validity, such that satisfiability and classical validity may be expressed by one and the same definition. Its dual[4] is adequate for rejectability and 'anti-validity'. Firstly, we need some preliminary definitions.

Definition 2.1 *The alphabet of CPL consists of*

(i)	*Propositional Variables*	$p_1, p_2, p_3, \ldots$
(ii)	*Logical Symbols*	$\neg, \wedge, \vee, \top, \bot$
(iii)	*Auxiliary Symbols*	$), ($.

$\mathcal{P}$ *denotes the set of propositional variables.*
$\mathcal{A}$ *denotes the set of atomic formulas,* i.e., $\mathcal{P} \cup \{\top, \bot\}$.
$\mathcal{L}$ *denotes the set of literals,* i.e., $\mathcal{P} \cup \{\neg\phi : \phi \in \mathcal{P}\}$.

Definition 2.2 *The set of formulas of CPL, denoted by $\mathcal{F}$, is the least set containing $\mathcal{A}$ that is closed under the operations $\neg, \wedge, \vee$.*

Definition 2.3 *Let $\Gamma \subseteq \mathcal{F}$ be a multiset, where $\emptyset$ denotes the empty multiset, and let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$. Then*

$$\text{(i)} \quad \bigvee\Gamma := \gamma_1 \vee \cdots \vee \gamma_k, \quad (\bigvee\emptyset := \top)$$
$$\text{(ii)} \quad \bigwedge\Gamma := \gamma_1 \wedge \cdots \wedge \gamma_k, \quad (\bigwedge\emptyset := \bot).$$

Definition 2.4 *The valuation function for a classical model M is defined as usual. For multisets $\Gamma \subseteq \mathcal{F}$ we define:*

$$\text{(i)} \quad V_M(\Gamma) = 1 := \forall\phi(\phi \in \Gamma \rightarrow V_M(\phi) = 1)$$
$$\text{(ii)} \quad V_M(\Gamma) = 0 := \forall\phi(\phi \in \Gamma \rightarrow V_M(\phi) = 0).$$

[3]For a synopsis of the history of theories of rejection for *CPL*, the reader may have recourse to Tamminga [7], which lacks a discussion of the recent system in Ishimoto [3].

[4]To the best of my knowledge, G. Stahl is the first logician who observed that in classical logic tautologies and contradictions are always each others duals. Cf. Stahl [6].

Definition 2.5 *Let $\Gamma \subseteq \mathcal{F}$ be a multiset and $\phi \in \mathcal{F}$ and M be a variable over classical models. Then*

$$\begin{array}{lll} \text{(i)} & \Gamma \models \phi & := \quad \forall M(V_M(\Gamma)=1 \rightarrow V_M(\phi)=1) \\ \text{(ii)} & \Gamma \rightleftharpoons \phi & := \quad \forall M(V_M(\Gamma)=0 \rightarrow V_M(\phi)=0). \end{array}$$

Using this standard notion of classical validity and its counterpart ('anti-validity'), we are able to define a more general notion of validity, which is needed to cover the notions of 'satisfiability' and 'rejectability'. I coin this new type of validity 'validity with respect to a literal basis', respectively 'anti-validity with respect to a literal basis'.

Definition 2.6 (Validity) *Let $\Gamma \subseteq \mathcal{F}$ be a multiset and $\phi \in \mathcal{F}$ and $\Delta \subseteq \mathcal{L}$ and M be a variable over classical models. Then*

$$\begin{array}{lll} \text{(i)} & \Delta;\Gamma \models^* \phi & := \quad \exists M(V_M(\Delta)=1) \text{ and } \Delta,\Gamma \models \phi \\ \text{(ii)} & \Delta;\Gamma \rightleftharpoons^* \phi & := \quad \exists M(V_M(\Delta)=0) \text{ and } \Delta,\Gamma \rightleftharpoons \phi. \end{array}$$

The following Lemma shows that validity with respect to a literal basis is more general than standard classical validity indeed, and that, in case the multiset of assumptions Γ is empty, our new definition implies that the formula ϕ is satisfiable (iii), or rejectable (iv). So, it is possible to uphold all the merits of the standard notion of classical validity, but at the same time having the semantical notion of satisfiability (rejectability) incorporated in the definition.

Lemma 2.7 *Let $\Gamma \subseteq \mathcal{F}$ be a multiset and $\phi \in \mathcal{F}$ and $\Delta \subseteq \mathcal{L}$. Then*

$$\begin{array}{llll} \text{(i)} & \emptyset;\Gamma \models^* \phi & \Longleftrightarrow & \Gamma \models \phi \\ \text{(ii)} & \emptyset;\Gamma \rightleftharpoons^* \phi & \Longleftrightarrow & \Gamma \rightleftharpoons \phi \\ \text{(iii)} & \Delta;\emptyset \models^* \phi & \Longrightarrow & \exists M(V_M(\phi)=1) \\ \text{(iv)} & \Delta;\emptyset \rightleftharpoons^* \phi & \Longrightarrow & \exists M(V_M(\phi)=0). \end{array}$$

Proof. This follows immediately from Definition 2.5 and Definition 2.6.•

3 Natural Deduction

In this section, a general Gentzen-style system of natural deduction—denoted by **ND**—will be defined. The system is general in the sense that it is written in a neutral notation, which allows for specific interpretations, exploiting the full symmetry between satisfiability and rejectability rules. The system uses uninterpreted 'variables' for logical symbols of *CPL*. Interpretations of these 'variables' give rise to two different subsystems: a system deriving satisfiable formulas and a system deriving rejectable formulas. Moreover, if we skip some of the axiom schemes which are available in these subsystems, we obtain two different subsubsystems: a system deriving tautologies and a system deriving contradictions.

In the remainder of the paper it will be shown that the general system **ND** covers the very same notions of validity as described above. I adopt the usual conventions governing the construction of proof-trees and the handling of assumptions in proof-trees. For a detailed exposition of these conventions, the reader may have recourse to Troelstra and Schwichtenberg [8], 20–22 and 29–34. Before we proceed to the definition of **ND**, we need an example to illustrate the preliminary definitions.

Let $\mathcal{D}_1$ be the following proof-tree, showing that $\{\neg p_1, \neg p_2\}; \emptyset \dashv^* \neg(p_1 \wedge p_2) \vee \neg p_2$. In other words, $\neg(p_1 \wedge p_2) \vee \neg p_2$ is anti-valid with respect to the literal basis $\{\neg p_1, \neg p_2\}$, as we shall see later on.

$$\dfrac{[p_1 \wedge p_2]^u \qquad \dfrac{\dfrac{[\![\neg p_1]\!] \quad [p_1]^v}{\top}\neg E \qquad \dfrac{[\![\neg p_2]\!] \quad [p_2]^w}{\top}\neg E}{}}{\dfrac{\dfrac{\top}{\neg(p_1 \wedge p_2)}\neg I_u \qquad [\![\neg p_2]\!]}{\neg(p_1 \wedge p_2) \vee \neg p_2}\vee I}\wedge E_{v,w}$$

An informal reading of what happens in this proof-tree is the following: If we assume that $p_1 \wedge p_2$ is false, then p_1 is false or p_2 is false. We can construct a model M in which $\neg p_1$ is false. Assume that p_1 is false in M. Then $\top$ is false in M. We can construct a model M' in which $\neg p_2$ is false. Assume that p_2 is false in M'. Then $\top$ is false in M'. As the models M and M' do not ascribe conflicting truth values to the propositional variables p_1 and p_2, there is a model M'' in which both $\neg p_1$ and $\neg p_2$ are false. In this model M'', the assumption that $p_1 \wedge p_2$ is false leads to a contradiction. Therefore, $\neg(p_1 \wedge p_2)$ is false in M''. Furthermore, we can construct a model M''' in which $\neg p_2$ is false. As the models M'' and M''' do not ascribe conflicting truth values to the propositional variables p_1 and p_2, there is a model M'''' in which both $\neg(p_1 \wedge p_2)$ and $\neg p_2$ are false. Then, in model M'''' the formula $\neg(p_1 \wedge p_2) \vee \neg p_2$ is false.

Let us now give a more formal definition, which will be used in the branching rules in the definition of the system **ND**, of jointly consistent sets of literals:

Definition 3.1 *Let $A \subseteq \mathcal{L}$ and $B \subseteq \mathcal{L}$. Then*

$$A \sim B, \text{ if } \neg\exists\phi(\{\phi, \neg\phi\} \subseteq A \cup B),$$

in other words, A and B are jointly consistent.

In the proof-tree above, the sets $\{\neg p_1\}$ and $\{\neg p_2\}$ are jointly consistent.

Let us recall that, in natural deduction trees, two assumptions $[\phi]^u$ and $[\psi]^v$ are identical iff both $\phi = \psi$ and $u = v$. Keeping this in mind, we define the following functions from proof-trees $\mathcal{D}$ to sets of formulas:

Definition 3.2 *Let $\mathcal{D}$ be a proof-tree in* **ND**. *Then*

(i) $[\![\mathcal{D}]\!]$ *is the set of all leafs of $\mathcal{D}$ of the form $[\![\phi]\!]$*
(ii) $[\mathcal{D}]$ *is the multiset of all different open assumptions of $\mathcal{D}$.*

In the example above, $[\![\mathcal{D}_1]\!] = \{\neg p_1, \neg p_2\}$ and $[\mathcal{D}_1] = \emptyset$, as all assumptions are closed. In the system **ND** defined below all proof-trees $\mathcal{D}$ lead to a (possibly empty) set of literals $[\![\mathcal{D}]\!]$.

Now, we give a definition of the system **ND**. Beforehand, we draw attention to the fact that branching rules can only be applied on the condition that the sets of axioms of the types (b) and (c), mentioned in Definition 3.3, used in the subtrees are jointly consistent. The additional axioms (b) and (c) allow us to construct literal by literal an auxiliary model, in which the derived formula, given the truth (respectively the falsity) of the open assumptions, is true (respectively false). In this way, satisfiability and rejectability can be modelled proof-theoretically.

Definition 3.3 *The system* **ND** *consists of exactly the following axiom schemes and rules:*

(i) *Axiom Schemes*

(a) $\oslash$
(b) $[\![\phi]\!]$, *if* $\phi \in \mathcal{P}$
(c) $[\![\neg\phi]\!]$, *if* $\phi \in \mathcal{P}$.

(ii) *Rules*

$$\dfrac{\begin{matrix}[\phi]^u\\ \mathcal{D}\\ \ominus\end{matrix}}{\neg\phi}\,\neg I_u \qquad \dfrac{\begin{matrix}[\neg\phi]^u\\ \mathcal{D}\\ \ominus\end{matrix}}{\phi}\,\ominus C_u \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \neg\phi\end{matrix}\quad \begin{matrix}\mathcal{D}_2\\ \phi\end{matrix}}{\ominus}\,\neg E,\ \ \textit{if}\ [\![\mathcal{D}_1]\!] \sim [\![\mathcal{D}_2]\!]$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \phi\oplus\psi\end{matrix}}{\phi}\,\oplus E_1 \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \phi\oplus\psi\end{matrix}}{\psi}\,\oplus E_2 \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \phi\end{matrix}\quad \begin{matrix}\mathcal{D}_2\\ \psi\end{matrix}}{\phi\oplus\psi}\,\oplus I,\ \ \textit{if}\ [\![\mathcal{D}_1]\!] \sim [\![\mathcal{D}_2]\!]$$

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \phi\otimes\psi\end{matrix}\quad \begin{matrix}[\phi]^u\\ \mathcal{D}_2\\ \chi\end{matrix}\quad \begin{matrix}[\psi]^v\\ \mathcal{D}_3\\ \chi\end{matrix}}{\chi}\,\otimes E_{u,v},\ \ \textit{if}\ [\![\mathcal{D}_1]\!] \sim [\![\mathcal{D}_2]\!] \sim [\![\mathcal{D}_3]\!] \sim [\![\mathcal{D}_1]\!]$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \phi\end{matrix}}{\phi\otimes\psi}\,\otimes I_1 \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \psi\end{matrix}}{\phi\otimes\psi}\,\otimes I_2$$

Substituting the usual logical constants $\top, \bot, \wedge, \vee$ in a systematical way for the uninterpreted variables $\oslash, \ominus, \oplus, \otimes$, we obtain the subsystems **NDT*** and **NDC***. The skipping of the axiom schemes (b) and (c) in these systems gives rise to the subsubsystems **NDT** and **NDC**:

Definition 3.4 *The systems* **NDT***, **NDT**, **NDC***, *and* **NDC** *are defined as follows:*

(i) **NDT*** := $\mathbf{ND}[{}^{\oslash}\!/_{\top}][{}^{\ominus}\!/_{\bot}][{}^{\oplus}\!/_{\wedge}][{}^{\otimes}\!/_{\vee}]$
(ii) **NDT** := **NDT*** *without axiom schemes* (b) *and* (c)
(iii) **NDC*** := $\mathbf{ND}[{}^{\oslash}\!/_{\bot}][{}^{\ominus}\!/_{\top}][{}^{\oplus}\!/_{\vee}][{}^{\otimes}\!/_{\wedge}]$
(iv) **NDC** := **NDC*** *without axiom schemes* (b) *and* (c).

Of course, if no axioms of the types (b) and (c) are availabe, then the conditions on $\neg E$, $\oplus I$, and $\otimes E$ become vacuous. I include the asterisk $*$ in the names for the systems and in the sign for derivability to indicate that the axioms enabling us to derive satisfiable and rejectable formulas are available in these systems. I coin this type of derivability as 'derivability with respect to a literal basis'. As we shall see later on, **NDT*** derives all satisfiable formulas, **NDT** derives all tautologies, **NDC*** derives all rejectable formulas, and **NDC** derives all contradictions.

Definition 3.5 *Let $\Gamma \subseteq \mathcal{F}$ be a multiset and $\phi \in \mathcal{F}$. Then*

(i)	$\Gamma \vdash \phi$	$:=$	*There exists a derivation $\mathcal{D}$ of ϕ from Γ in* **NDT**
(ii)	$\Delta; \Gamma \vdash^* \phi$	$:=$	*There exists a derivation $\mathcal{D}$ of ϕ from Γ in* **NDT***, *where $\Delta = [\![\mathcal{D}]\!]$*
(iii)	$\Delta; \Gamma \dashv^* \phi$	$:=$	*There exists a derivation $\mathcal{D}$ of ϕ from Γ in* **NDC***, *where $\Delta = [\![\mathcal{D}]\!]$*
(iv)	$\Gamma \dashv \phi$	$:=$	*There exists a derivation $\mathcal{D}$ of ϕ from Γ in* **NDC**.

Lemma 3.6 *Let $\Gamma \subseteq \mathcal{F}$ be a multiset and $\phi \in \mathcal{F}$. Then*

$$\begin{array}{ll} \text{(i)} & \Gamma \vdash \phi \iff \emptyset; \Gamma \vdash^* \phi \\ \text{(ii)} & \Gamma \dashv \phi \iff \emptyset; \Gamma \dashv^* \phi. \end{array}$$

Proof. This follows immediately from Definition 3.4 and Definition 3.5.•

4 Correctness of ND

The informal explanation of the proof-tree just before Definition 3.1 already contains an argument for the correctness of the used rules. This argument will be formalized in the following Lemma, covering all the rules of **NDT***. Because of its facility, the proof is left to the reader.

Lemma 4.1 *Let $\Gamma_j \subseteq \mathcal{F}$ be multisets for $j \in \{1,2,3\}$ and $\phi, \psi, \chi \in \mathcal{F}$ and $\Delta_i \subseteq \mathcal{L}$ for $i \in \{1,2,3\}$. Then*

$$\begin{array}{rlll}
\text{(i)} & \Delta; \Gamma \models^* \phi, & \textit{if} & \phi \in \Delta \textit{ and } \exists M (V_M(\Delta) = 1) \\
\text{(ii)} & \Delta; \Gamma \models^* \phi, & \textit{if} & \phi \in \Gamma \textit{ and } \exists M (V_M(\Delta) = 1) \\
\text{(iii)} & \Delta; \Gamma \models^* \top, & \textit{if} & \exists M (V_M(\Delta) = 1) \\
\text{(iv)} & \Delta; \Gamma, \phi \models^* \bot & \Longrightarrow & \Delta; \Gamma \models^* \neg\phi \\
\text{(v)} & \Delta; \Gamma, \neg\phi \models^* \bot & \Longrightarrow & \Delta; \Gamma \models^* \phi \\
\text{(vi)} & \left.\begin{array}{l} \Delta_1; \Gamma_1 \models^* \neg\phi \\ \Delta_2; \Gamma_2 \models^* \phi \\ \Delta_1 \sim \Delta_2 \end{array}\right\} & \Longrightarrow & \Delta_1, \Delta_2; \Gamma_1, \Gamma_2 \models^* \bot \\
\text{(vii)} & \Delta; \Gamma \models^* \phi \wedge \psi & \Longrightarrow & \Delta; \Gamma \models^* \phi \\
\text{(viii)} & \Delta; \Gamma \models^* \phi \wedge \psi & \Longrightarrow & \Delta; \Gamma \models^* \psi \\
\text{(ix)} & \left.\begin{array}{l} \Delta_1; \Gamma_1 \models^* \phi \\ \Delta_2; \Gamma_2 \models^* \psi \\ \Delta_1 \sim \Delta_2 \end{array}\right\} & \Longrightarrow & \Delta_1, \Delta_2; \Gamma_1, \Gamma_2 \models^* \phi \wedge \psi \\
\text{(x)} & \left.\begin{array}{l} \Delta_1; \Gamma_1 \models^* \phi \vee \psi \\ \Delta_2; \Gamma_2, \phi \models^* \chi \\ \Delta_3; \Gamma_3, \psi \models^* \chi \\ \Delta_1 \sim \Delta_2 \sim \Delta_3 \sim \Delta_1 \end{array}\right\} & \Longrightarrow & \Delta_1, \Delta_2, \Delta_3; \Gamma_1, \Gamma_2, \Gamma_3 \models^* \chi \\
\text{(xi)} & \Delta; \Gamma \models^* \phi & \Longrightarrow & \Delta; \Gamma \models^* \phi \vee \psi \\
\text{(xii)} & \Delta; \Gamma \models^* \psi & \Longrightarrow & \Delta; \Gamma \models^* \phi \vee \psi.
\end{array}$$

Using Lemma 4.1, we can prove the correctness of **NDT*** by a routine induction on the depth of the derivation $\mathcal{D}$ for ϕ from Γ. If we prove the analogue of Lemma 4.1 for $\models\!\!\!\dashv^*$, we can prove the correctness of **NDC*** in the same way. This gives us the following theorem:

Theorem 4.2 (Correctness of NDT* and NDC*) *Let $\Gamma \subseteq \mathcal{F}$ be a multiset and $\phi \in \mathcal{F}$ and $\Delta \subseteq \mathcal{L}$. Then*

$$\begin{array}{lll} \text{(i)} & \Delta;\Gamma \vdash^{*} \phi & \Longrightarrow \quad \Delta;\Gamma \models^{*} \phi \\ \text{(ii)} & \Delta;\Gamma \dashv^{*} \phi & \Longrightarrow \quad \Delta;\Gamma \dashv\!\mid^{*} \phi. \end{array}$$

As a corollary, of course, the systems **NDT** and **NCC**, which are special cases of **NDT*** and **NDC***, are correct:

Corollary 4.3 (Correctness of NDT and NCC) *Let $\Gamma \subseteq \mathcal{F}$ be a multiset and $\phi \in \mathcal{F}$. Then*

$$\begin{array}{lll} \text{(i)} & \Gamma \vdash \phi & \Longrightarrow \quad \Gamma \models \phi \\ \text{(ii)} & \Gamma \dashv \phi & \Longrightarrow \quad \Gamma \dashv\!\mid \phi. \end{array}$$

Proof. This follows immediately from Lemma 3.6, Lemma 2.7(i)(ii), and Theorem 4.2.•

5 Sequent Systems

In this section a general Gentzen-style sequent calculus—denoted by **SC**—shall be set forth. I conform to the usual conventions governing the construction of proof-trees for sequent systems. A detailed exposition of these conventions can be found in Troelstra and Schwichtenberg [8], 23–25, 51–58, and 65–71. In the remainder of this section Γ and Σ refer to multisets, whereas Δ refers to sets. Both the empty set and the empty multiset are represented by $\emptyset$, but the context gives the symbol an unambiguous interpretation: only in the context of literal bases (*i.e.*, on the left of the semicolon), $\emptyset$ denotes the empty set.

Definition 5.1 *The system* **SC** *consists of exactly the following axiom schemes and rules:*

(i) *Axiom Schemes*

(a) $\emptyset;\phi \Rightarrow \phi$
(b) $\emptyset;\ominus \Rightarrow \emptyset$
(c) $\emptyset;\emptyset \Rightarrow \oslash$
(d) $\Delta;\emptyset \Rightarrow \emptyset$ *if Δ is a consistent set of literals.*

(ii) *Structural Rules*

$$\frac{\Delta;\Gamma\Rightarrow\Sigma}{\Delta;\Gamma,\phi\Rightarrow\Sigma}\ L\Delta,\quad \textit{if } \neg\phi\in\Delta \qquad\qquad \frac{\Delta;\Gamma\Rightarrow\Sigma}{\Delta;\Gamma\Rightarrow\Sigma,\phi}\ R\Delta,\quad \textit{if } \phi\in\Delta$$

$$\frac{\Delta;\Gamma\Rightarrow\Sigma}{\Delta;\Gamma,\phi\Rightarrow\Sigma}\ LW,\quad \textit{if } \Gamma\cup\Sigma\neq\emptyset \qquad\qquad \frac{\Delta;\Gamma\Rightarrow\Sigma}{\Delta;\Gamma\Rightarrow\Sigma,\phi}\ RW,\quad \textit{if } \Gamma\cup\Sigma\neq\emptyset$$

$$\frac{\Delta;\Gamma,\phi,\phi\Rightarrow\Sigma}{\Delta;\Gamma,\phi\Rightarrow\Sigma}\ LC \qquad\qquad \frac{\Delta;\Gamma\Rightarrow\Sigma,\phi,\phi}{\Delta;\Gamma\Rightarrow\Sigma,\phi}\ RC$$

$$\frac{\Delta_1;\Gamma_1\Rightarrow\Sigma_1 \qquad \Delta_2;\Gamma_2\Rightarrow\Sigma_2}{\Delta_1,\Delta_2;\Gamma_1,\Gamma_2\Rightarrow\Sigma_1,\Sigma_2}\ \textit{Merge},\quad \textit{if } \Delta_1\sim\Delta_2$$

(iii) *Logical Rules*

$$\frac{\Delta;\Gamma\Rightarrow\Sigma,\phi}{\Delta;\Gamma,\neg\phi\Rightarrow\Sigma}\,L\,\neg \qquad \frac{\Delta;\Gamma,\phi\Rightarrow\Sigma}{\Delta;\Gamma\Rightarrow\Sigma,\neg\phi}\,R\,\neg$$

$$\frac{\Delta;\Gamma,\phi_i\Rightarrow\Sigma}{\Delta;\Gamma,\phi_0\oplus\phi_1\Rightarrow\Sigma}\,L\,\oplus,\quad i=(0,1)$$

$$\frac{\Delta_1;\Gamma\Rightarrow\Sigma,\phi \qquad \Delta_2;\Gamma\Rightarrow\Sigma,\psi}{\Delta_1,\Delta_2;\Gamma\Rightarrow\Sigma,\phi\oplus\psi}\,R\,\oplus,\quad \textit{if}\ \Delta_1\sim\Delta_2$$

$$\frac{\Delta_1;\Gamma,\phi\Rightarrow\Sigma \qquad \Delta_2;\Gamma,\psi\Rightarrow\Sigma}{\Delta_1,\Delta_2;\Gamma,\phi\otimes\psi\Rightarrow\Sigma}\,L\,\otimes,\quad \textit{if}\ \Delta_1\sim\Delta_2$$

$$\frac{\Delta;\Gamma\Rightarrow\Sigma,\phi_i}{\Delta;\Gamma\Rightarrow\Sigma,\phi_0\otimes\phi_1}\,R\,\otimes,\quad i=(0,1)$$

Substituting the usual logical constants $\top,\bot,\wedge,\vee$ in a systematical way for the uninterpreted variables $\oslash,\ominus,\oplus,\otimes$, we obtain the subsystems **SCT*** and **SCC***. The skipping of the axiom scheme (d) in these systems gives rise to the subsubsystems **SCT** and **SCC**:

Definition 5.2 *The systems* **SCT***, **SCT**, **SCC***, *and* **SCC** *are defined as follows:*

(i)	**SCT***	:=	$\mathbf{SC}[^{\oslash}/_{\top}][^{\ominus}/_{\bot}][^{\oplus}/_{\wedge}][^{\otimes}/_{\vee}]$
(ii)	**SCT**	:=	**SCT*** *without axiom scheme* (d)
(iii)	**SCC***	:=	$\mathbf{SC}[^{\oslash}/_{\bot}][^{\ominus}/_{\top}][^{\oplus}/_{\vee}][^{\otimes}/_{\wedge}]$
(iv)	**SCC**	:=	**SCC*** *without axiom scheme* (d).

Of course, if no axioms of the type (d) are availabe, then the conditions on *LW*, *RW*, $R\,\oplus$, and $L\,\otimes$ become vacuous, whilst LΔ and RΔ become useless. Note that **SCT** is a standard Gentzen system for *CPL*, as the *Merge* rule follows from the weakening rules of **SCT**.

Once again, derivability with respect to a literal basis will be defined, now with respect to the sequent systems previously defined.

Definition 5.3 *Let* $\Gamma,\Sigma\subseteq\mathcal{F}$ *be multisets and let* $\Delta\subseteq\mathcal{L}$. *Then*

(i)	$\vdash\Gamma\Rightarrow\Sigma$	:=	*There exists a derivation* $\mathcal{D}$ *of* $\emptyset;\Gamma\Rightarrow\Sigma$ *in* **SCT**
(ii)	$\vdash^*\Delta;\Gamma\Rightarrow\Sigma$	:=	*There exists a derivation* $\mathcal{D}$ *of* $\Delta;\Gamma\Rightarrow\Sigma$ *in* **SCT***
(iii)	$\dashv^*\Delta;\Gamma\Rightarrow\Sigma$	:=	*There exists a derivation* $\mathcal{D}$ *of* $\Delta;\Gamma\Rightarrow\Sigma$ *in* **SCC***
(iv)	$\dashv\Gamma\Rightarrow\Sigma$	:=	*There exists a derivation* $\mathcal{D}$ *of* $\emptyset;\Gamma\Rightarrow\Sigma$ *in* **SCC**.

Derivability in **SCT** and **SCC** is, of course, equivalent to derivability with respect to an *empty* literal basis in **SCT*** and **SCC***, respectively. Stated formally:

Lemma 5.4 *Let* $\Gamma,\Sigma\subseteq\mathcal{F}$ *be multisets. Then*

(i) $\vdash\Gamma\Rightarrow\Sigma \iff \vdash^*\emptyset;\Gamma\Rightarrow\Sigma$

(ii) $\dashv\Gamma\Rightarrow\Sigma \iff \dashv^*\emptyset;\Gamma\Rightarrow\Sigma$.

Proof. This follows immediately from Definition 5.2 and Definition 5.3.•

6 From SC to ND

In this section, we shall prove that for every **SC**-derivation there is a corresponding **ND**-derivation. The proof is by a quite straightforward induction on the depth of the **SC**-derivation.

Theorem 6.1 *Let* $\Gamma, \Sigma \subseteq \mathcal{F}$ *be multisets and* $\Delta \subseteq \mathcal{L}$. *Then*

$$\begin{array}{ll} \text{(i)} & \vdash^* \Delta;\Gamma \Rightarrow \Sigma \implies \Delta;\Gamma \vdash^* \bigvee \Sigma \\ \text{(ii)} & \dashv^* \Delta;\Gamma \Rightarrow \Sigma \implies \Delta;\Gamma \dashv^* \bigwedge \Sigma. \end{array}$$

Proof. (i) By induction on the depth n of the derivation of $\vdash^* \Delta;\Gamma \Rightarrow \Sigma$.

Basis: $n=1$. Then $\vdash^* \Delta;\Gamma \Rightarrow \Sigma$ is an axiom of the type (a), (b), (c), or (d). As the other cases are trivial, let us consider case (d). Then we must have the axiom $\Delta;\emptyset \Rightarrow \emptyset$. According to Definition 2.3 we have $\bigvee \emptyset = \top$, the following derivation scheme—which must be repeated for every element in Δ—does the job (let us assume that $\phi \in \Delta$):

$$\dfrac{\dfrac{\top \quad [\![\phi]\!]}{\top \wedge \phi}}{\top} \qquad \textit{with } [\![\mathcal{D}]\!] = \{\phi\} \textit{ and } [\mathcal{D}] = \emptyset.$$

Induction Hypothesis: Let the proposition to be proved be correct for $n-1$.

Induction Step: Now, we have to split cases according to the last derivation rule which was used to derive $\Delta;\Gamma \Rightarrow \Sigma$. Leaving the other cases aside, we shall prove the most difficult case: $R\wedge$.

(Case $R\wedge$) It is given that $\vdash^* \Delta;\Gamma \Rightarrow \Sigma$. Let $\Sigma = \Sigma' \cup \{\phi \wedge \psi\}$. Hence, there must be a derivation for the following sequent:

$$\Delta;\Gamma \Rightarrow \Sigma', \phi\wedge\psi,$$

which has been constructed by $R\wedge$ from the sequents:

$$\Delta_1;\Gamma \Rightarrow \Sigma', \phi \quad \textit{and} \quad \Delta_2;\Gamma \Rightarrow \Sigma', \psi,$$

such that $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta_1 \sim \Delta_2$. Because of the induction hypothesis, the following assertions are true:

$$\begin{array}{ll} (1) & \Delta_1;\Gamma \vdash^* \bigvee \Sigma' \vee \phi \\ (2) & \Delta_2;\Gamma \vdash^* \bigvee \Sigma' \vee \psi \\ (3) & \Delta_1 \sim \Delta_2, \end{array}$$

which means there are derivations $\mathcal{D}_1$ and $\mathcal{D}_2$, such that

$$\begin{array}{c} \mathcal{D}_1 \\ \bigvee \Sigma' \vee \phi \end{array} \qquad \textit{with } [\![\mathcal{D}_1]\!] = \Delta_1 \textit{ and } [\mathcal{D}_1] = \Gamma,$$

and

$$\begin{array}{c} \mathcal{D}_2 \\ \bigvee \Sigma' \vee \psi \end{array} \qquad \textit{with } [\![\mathcal{D}_2]\!] = \Delta_2 \textit{ and } [\mathcal{D}_2] = \Gamma.$$

Derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ can be extended to the following derivation $\mathcal{D}^*$ (the correctness of the merging of these proof-trees is guaranteed by $[\![\mathcal{D}_1]\!] \sim [\![\mathcal{D}_2]\!]$):

$$
\dfrac{\dfrac{\mathcal{D}_1}{\bigvee\Sigma'\vee\phi} \qquad \dfrac{[\bigvee\Sigma']^u}{\bigvee\Sigma'\vee(\phi\wedge\psi)} \qquad \dfrac{\dfrac{\mathcal{D}_2}{\bigvee\Sigma'\vee\psi} \qquad \dfrac{[\bigvee\Sigma']^w}{\bigvee\Sigma'\vee(\phi\wedge\psi)} \qquad \dfrac{\dfrac{[\phi]^v \quad [\psi]^x}{\phi\wedge\psi}}{\bigvee\Sigma'\vee(\phi\wedge\psi)}}{\bigvee\Sigma'\vee(\phi\wedge\psi)}\, w,x}{\bigvee\Sigma'\vee(\phi\wedge\psi)}\, u,v
$$

with $[\![\mathcal{D}^*]\!]=\Delta$ *and* $[\mathcal{D}^*]=\Gamma$.

This implies the desired conclusion: $\Delta;\Gamma\vdash^*\bigvee\Sigma$.

The procedure of proving (ii) is completely analogous to (i).•

7 Completeness of SC

In the remainder of this paper we shall prove that all systems discussed so far are complete with respect to their intended semantics. To close the circle of theorems needed, we have to prove the following theorem. In the proof of the theorem we adapt a method of proving completeness of sequent calculi described by Heindorf [2], 83–90.

Theorem 7.1 *Let* $\Gamma,\Sigma\subseteq\mathcal{F}$ *be multisets and* $\Delta\subseteq\mathcal{L}$. *Then*

$$
\begin{array}{lllll}
\text{(i)} & \Delta;\Gamma\models^*\bigvee\Sigma & \Longrightarrow & \vdash^*\Delta;\Gamma\Rightarrow\Sigma \\
\text{(ii)} & \Delta;\Gamma\mathrel{=\!\!|}^*\bigvee\Sigma & \Longrightarrow & \dashv^*\Delta;\Gamma\Rightarrow\Sigma.
\end{array}
$$

Proof. I shall prove the converse of (i). Let it be given that $\nvdash^*\Delta;\Gamma\Rightarrow\Sigma$. First, we fix the order of the given sequent: In the course of this proof Γ and Σ and their correlates shall refer to *ordered* multisets. If Δ is inconsistent, then we are done, as $\exists_M(V_M(\Delta)=1)$ turns out to be false. Let us suppose, then, that Δ is consistent. Now, we have to make sure that $\Delta;\Gamma\not\models^*\Sigma$, that is $\exists_M(V_M(\Delta,\Gamma)=1\wedge V_M(\bigvee\Sigma)=0)$, because $\exists_M(V_M(\Delta)=1)$. We shall build such a model with the aid of an infinite sequence of sequents, which is defined inductively as follows:

$n=0$	$\Delta;\Gamma_0\Rightarrow\Sigma_0$	$:=$	$\Delta;\Gamma\Rightarrow\Sigma$
$n>0$	$\Delta;\Gamma_n\Rightarrow\Sigma_n$		$\Delta;\Gamma_{n+1}\Rightarrow\Sigma_{n+1}$
n *odd*	$\Delta;\emptyset\Rightarrow\Sigma_n$	$\mapsto$	$\Delta;\emptyset\Rightarrow\Sigma_n$
	$\Delta;\Gamma',\phi\Rightarrow\Sigma_n$ *and* $\phi\in\mathcal{A}$	$\mapsto$	$\Delta;\phi,\Gamma'\Rightarrow\Sigma_n$
	$\Delta;\Gamma',\neg\phi\Rightarrow\Sigma_n$	$\mapsto$	$\Delta;\neg\phi,\Gamma'\Rightarrow\Sigma_n,\phi$
	$\Delta;\Gamma',\phi\wedge\psi\Rightarrow\Sigma_n$	$\mapsto$	$\Delta;\phi\wedge\psi,\phi,\psi,\Gamma'\Rightarrow\Sigma_n$
	$\Delta;\Gamma',\phi\vee\psi\Rightarrow\Sigma_n$	$\mapsto$	$\Delta;\phi\vee\psi,\phi,\Gamma'\Rightarrow\Sigma_n$ *or* $\Delta;\phi\vee\psi,\psi,\Gamma'\Rightarrow\Sigma_n$
n *even*	$\Delta;\Gamma_n\Rightarrow\emptyset$	$\mapsto$	$\Delta;\Gamma_n\Rightarrow\emptyset$
	$\Delta;\Gamma_n\Rightarrow\phi,\Sigma'$ *and* $\phi\in\mathcal{A}$	$\mapsto$	$\Delta;\Gamma_n\Rightarrow\Sigma',\phi$
	$\Delta;\Gamma_n\Rightarrow\neg\phi,\Sigma'$	$\mapsto$	$\Delta;\phi,\Gamma_n\Rightarrow\Sigma',\neg\phi$
	$\Delta;\Gamma_n\Rightarrow\phi\wedge\psi,\Sigma'$	$\mapsto$	$\Delta;\Gamma_n\Rightarrow\Sigma',\phi,\phi\wedge\psi$ *or* $\Delta;\Gamma_n\Rightarrow\Sigma',\psi,\phi\wedge\psi$
	$\Delta;\Gamma_n\Rightarrow\phi\vee\psi,\Sigma'$	$\mapsto$	$\Delta;\Gamma_n\Rightarrow\Sigma',\phi,\psi,\phi\vee\psi$

The reader may easily show that $\nvdash^*\Delta;\Gamma_{n+1}\Rightarrow\Sigma_{n+1}$. In case an alternative is offered, this must be true for at least one of the alternatives. Choose the alternative for which this is true. We also can make the following observations:

(α) If $\phi\in\Gamma_k$ ($\phi\in\Sigma_k$), then for all n with $n\geq k$ holds $\phi\in\Gamma_n$ ($\phi\in\Sigma_n$).

(β) If $\phi \in \Gamma_k$ ($\phi \in \Sigma_k$), then there is an odd (even) n with $n \geq k$ such that ϕ appears as last (first) formula in Γ_n (Σ_n).

Let $\Gamma_\infty = \bigcup_{n=0}^{\infty} \Gamma_n$ and $\Sigma_\infty = \bigcup_{n=0}^{\infty} \Sigma_n$. Now, we can prove the following assertions to be true:

(γ) $(\Delta \cup \Gamma_\infty) \cap \Sigma_\infty = \emptyset$.

(δ) $\Delta \cup (\Gamma_\infty \cap \mathcal{L})$ is consistent.

In order to prove assertion (γ), let us suppose that $(\Delta \cup \Gamma_\infty) \cap \Sigma_\infty \neq \emptyset$. Then there must be a ϕ such that (a) $\phi \in (\Gamma_\infty \cap \Sigma_\infty)$ or (b) $\phi \in (\Delta \cap \Sigma_\infty)$.

Suppose that (a). Then there is a sequent $\Delta; \Gamma_n \Rightarrow \Sigma_n$ such that $\phi \in \Gamma_n$ and $\phi \in \Sigma_n$. But then we can construct the following derivation, which is impossible according to what we proved above (double lines indicate some (possibly zero) applications of the structural rules *LW* and *RW*):

$$\dfrac{\Delta; \emptyset \Rightarrow \emptyset \qquad \dfrac{\emptyset; \phi \Rightarrow \phi}{\emptyset; \Gamma_n \Rightarrow \Sigma_n}}{\Delta; \Gamma_n \Rightarrow \Sigma_n}\ \mathit{Merge}$$

Suppose that (b). Then there is a sequent $\Delta; \Gamma_n \Rightarrow \Sigma_n$ such that $\phi \in \Delta$ and $\phi \in \Sigma_n$. We have either $\phi = \psi$ where $\psi \in \mathcal{P}$ or $\phi = \neg\psi$ where $\psi \in \mathcal{P}$. But then we can construct one of the following derivations, either of which is impossible according to what we proved above:

$$\dfrac{\dfrac{\Delta; \emptyset \Rightarrow \emptyset}{\Delta; \emptyset \Rightarrow \psi}\ R\Delta}{\Delta; \Gamma_n \Rightarrow \Sigma_n} \qquad\qquad \dfrac{\dfrac{\dfrac{\Delta; \emptyset \Rightarrow \emptyset}{\Delta; \psi \Rightarrow \emptyset}\ L\Delta}{\Delta; \emptyset \Rightarrow \neg\psi}\ R\neg}{\Delta; \Gamma_n \Rightarrow \Sigma_n}$$

Therefore, $(\Delta \cup \Gamma_\infty) \cap \Sigma_\infty = \emptyset$.

In order to prove assertion (δ), let us suppose that $\Delta \cup (\Gamma_\infty \cap \mathcal{L})$ is inconsistent. Then there must be a ϕ such that (a) $\phi \in \Delta$ and $\neg\phi \in (\Gamma_\infty \cap \mathcal{L})$ or (b) $\neg\phi \in \Delta$ and $\phi \in (\Gamma_\infty \cap \mathcal{L})$.

Suppose that (a) Then there is a sequent $\Delta; \Gamma_n \Rightarrow \Sigma_n$ such that $\phi \in \Delta$ and $\neg\phi \in \Gamma_n$ and $\phi \in \mathcal{P}$. But then we can construct the following derivation, which is impossible according to what we proved above:

$$\dfrac{\dfrac{\dfrac{\Delta; \emptyset \Rightarrow \emptyset}{\Delta; \emptyset \Rightarrow \phi}\ R\Delta}{\Delta; \neg\phi \Rightarrow \emptyset}\ L\neg}{\Delta; \Gamma_n \Rightarrow \Sigma_n}$$

Suppose that (b) Then there is a sequent $\Delta; \Gamma_n \Rightarrow \Sigma_n$ such that $\neg\phi \in \Delta$ and $\phi \in \Gamma_n$ and $\phi \in \mathcal{P}$. But then we can construct the following derivation, which is impossible according to what we proved above:

$$\dfrac{\dfrac{\Delta; \emptyset \Rightarrow \emptyset}{\Delta; \phi \Rightarrow \emptyset}\ L\Delta}{\Delta; \Gamma_n \Rightarrow \Sigma_n}$$

Therefore, $\Delta \cup (\Gamma_\infty \cap \mathcal{L})$ is consistent.

Right now, we define a classical model M, the existence of which is guaranteed by (γ) and (δ), in the following way: $V_M((\Delta \cup \Gamma_\infty) \cap \mathcal{P}) = 1$ and $V_M((\Delta \cup \Sigma_\infty) \cap \mathcal{P}) = 0$. This model has the following desired characteristics:

(ϵ) If $\phi \in \Gamma_\infty$, then $V_M(\phi) = 1$.

(ζ) If $\phi \in \Sigma_\infty$, then $V_M(\phi) = 0$.

A proof of (ϵ) and (ζ) is obtained by induction on the number of logical operators of ϕ, using (β) and the table indicating the rules of construction of $\Delta; \Gamma_{n+1} \Rightarrow \Sigma_{n+1}$ from $\Delta; \Gamma_n \Rightarrow \Sigma_n$. The basis of the induction is correct by definition of M. Let us suppose that the assertion to be proved is correct for every ψ with less logical operators than ϕ. Now, we split cases according to the principal operator of ϕ. Leaving all other cases aside, we shall consider the case where $\phi = \psi_1 \vee \psi_2$ and $\psi_1 \vee \psi_2 \in \Gamma_\infty$. According to (β), there is a n such that $\Delta; \Gamma'_n, \psi_1 \vee \psi_2 \Rightarrow \Sigma_n$, where $\Gamma_n = \Gamma'_n, \psi_1 \vee \psi_2$. Therefore, $\psi_1 \in \Gamma_{n+1}$ or $\psi_2 \in \Gamma_{n+1}$. By our induction hypothesis, we have $V_M(\psi_1) = 1$ or $V_M(\psi_2) = 1$. Hence $V_M(\psi_1 \vee \psi_2) = 1$.

Therefore, M is a 'witness' (in the technical sense of the word) of the following claim: $\exists M(V_M(\Delta)=1)$ and $\exists M(V_M(\Delta, \Gamma)=1 \wedge V_M(\Sigma)=0)$. Hence, $\Delta; \Gamma \not\models^* \Sigma$.

The procedure of proving (ii) is completely analogous to (i).•

As the preceding theorem closes the circle of theorems, let us summarize the results of this and the previous sections in the following theorem:

Theorem 7.2 *Let $\Gamma \subseteq \mathcal{F}$ be a multiset and $\phi \in \mathcal{F}$ and $\Delta \subseteq \mathcal{L}$. Then*

$$\begin{array}{llllll} \text{(i)} & \Delta; \Gamma \vdash^* \phi & \Longleftrightarrow & \vdash^* \Delta; \Gamma \Rightarrow \phi & \Longleftrightarrow & \Delta; \Gamma \models^* \phi \\ \text{(ii)} & \Delta; \Gamma \dashv^* \phi & \Longleftrightarrow & \dashv^* \Delta; \Gamma \Rightarrow \phi & \Longleftrightarrow & \Delta; \Gamma \;=\!\!|^* \phi. \end{array}$$

It turns out to be the case that the usual notions of 'validity' and 'derivability' are special cases of the more general notions of 'validity with respect to a literal basis' and 'derivability with respect to a literal basis' which have been defined in this paper. As a corollary of the previous theorem, we can, putting $\Delta = \emptyset$, state the following well-known theorem:

Corollary 7.3 *Let $\Gamma \subseteq \mathcal{F}$ be a multiset and $\phi \in \mathcal{F}$. Then*

$$\begin{array}{llllll} \text{(i)} & \Gamma \vdash \phi & \Longleftrightarrow & \vdash \Gamma \Rightarrow \phi & \Longleftrightarrow & \Gamma \models \phi \\ \text{(ii)} & \Gamma \dashv \phi & \Longleftrightarrow & \dashv \Gamma \Rightarrow \phi & \Longleftrightarrow & \Gamma \;=\!\!|\; \phi. \end{array}$$

8 Conclusion

The results in this paper show that, contrary to the existing literature on logics of rejection, it is possible to stick to the standard proof-theoretical structures in deriving all non-theorems of *CPL*. Moreover, not only are tautologies and contradictions each other's duals, but this also holds for satisfiable and rejectable formulas. Surprisingly, this duality can be extended to the metatheory in which the proofs about the systems are formulated.

Moreover, I would like to draw attention once again to the fact that in the systems **NDT*** and **NDC*** the construction of a proof involves the building of an auxiliary model, the consistency of which is guaranteed by the conditions on the application of the branching rules. In this way, semantical concepts like 'satisfiability' and 'rejectability' can be modelled proof-theoretically.

There is a close and interesting connection between the system proving satisfiability described above and Veltman's system for *CPL*, including an operator 'might' in the language, presented in [9], 227–231. Given the consistency of a set of beliefs Γ, Veltman develops a logic characterizing the statements which might be true, *i.e.*, the statements which are satisfiable, given the truth of the set of beliefs Γ. To map these ideas into the systems presented in this paper, an additional constraint on the use of literals must be formulated: the set Δ of literals used in the derivations must be consistent with the set of beliefs Γ. It should be clear from the start that there may be jointly inconsistent sets Δ_1 and Δ_2 such that both $\Delta_1 \cup \Gamma$ and $\Delta_2 \cup \Gamma$ are consistent. At any rate, this poses some questions for future research.

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